This talk (Google Slides):
https://tinyurl.com/icermzach

BlackHoles@Home Status Report

Zach Etienne

Center for Gravitational Waves And Cosmology

NRPy+: Python-based C code generation framework for NR

Tensorial expressions in Einstein-like notation ⇒ Highly optimized C-code kernels (with FDs)

"Nerpy", the NRPy+ mascot. Photo CC2.0 Pacific Environment (modified).

https://nrpyplus.net

https://github.com/zachetienne/nrpytutorial

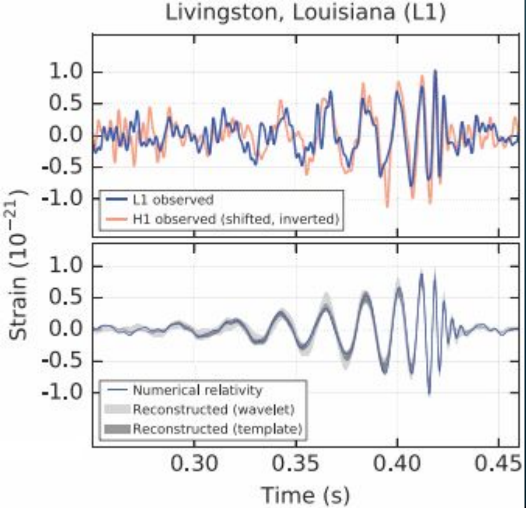
BlackHoles@Home

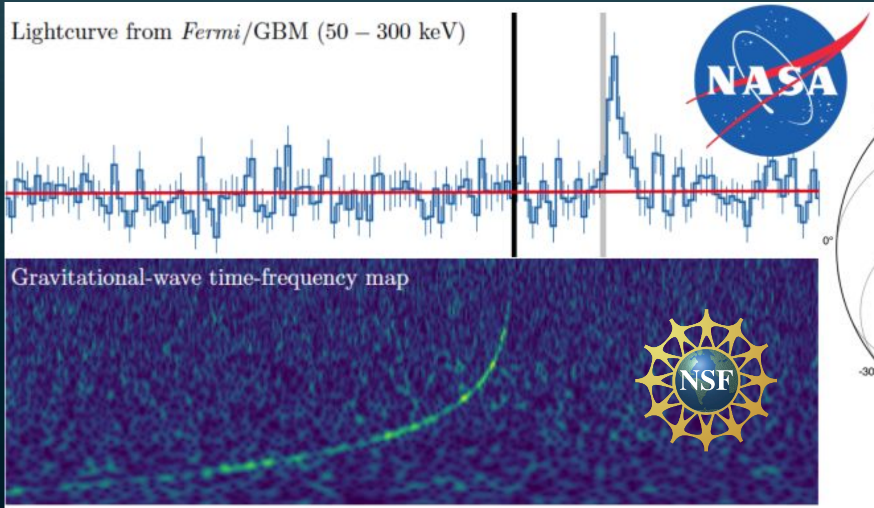
https://blackholesathome.net

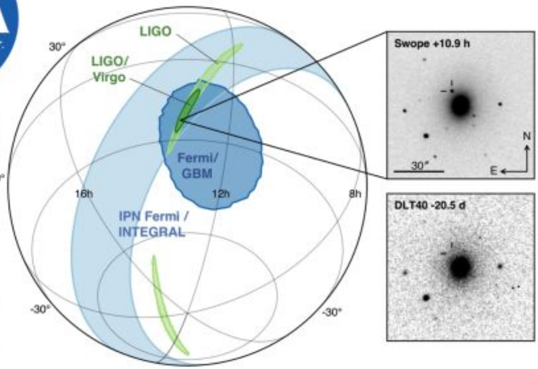
BlackHoles@Home aims to fit numerical-relativity-based binary black hole (BBH) calculations on a consumer-grade desktop computer, enabling gravitational waveform follow-ups and catalogs at unprecedentedly large scales using volunteer computers.

Importance of modeling gravitational wave and multimessenger sources

- Example: LIGO detects a gravitational wave from a black hole or neutron star binary

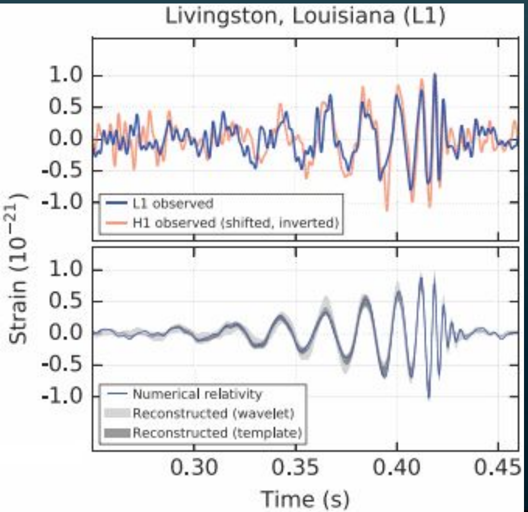


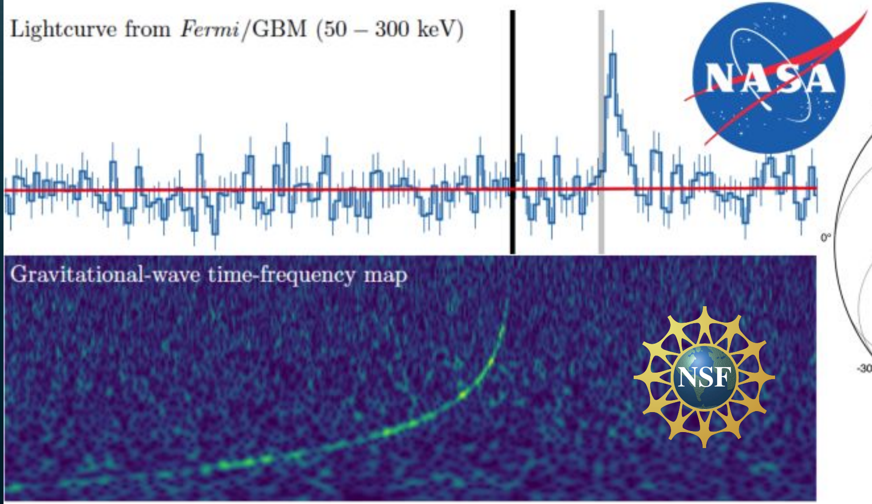


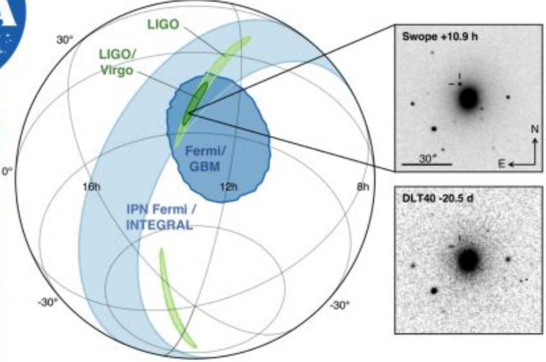


Importance of modeling gravitational wave and multimessenger sources

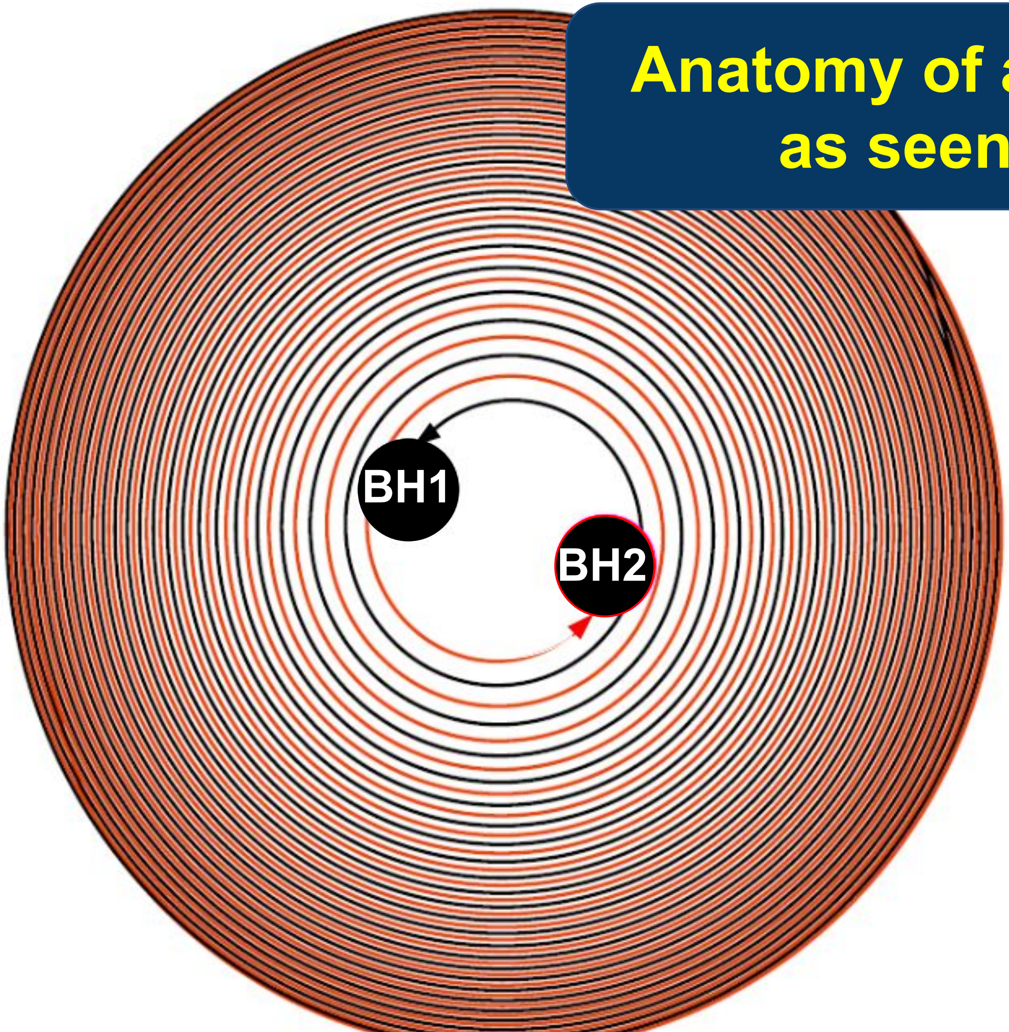
- $1B+ Question: What *exactly* caused this and *how*?
  - Answer can provide deep insights into extreme gravity & extreme matter, testing theories beyond current limits
  - To advance science, must compare observations with theoretical predictions
    - Theoretical predictions need to span observ. & theor. uncertainties







Anatomy of a Binary Black Hole Merger, as seen in gravitational waves



Lovelace et al., CQG 29, **045003** (2012) (modified)

- Gravitational-wave driven "Relativistic death spiral"

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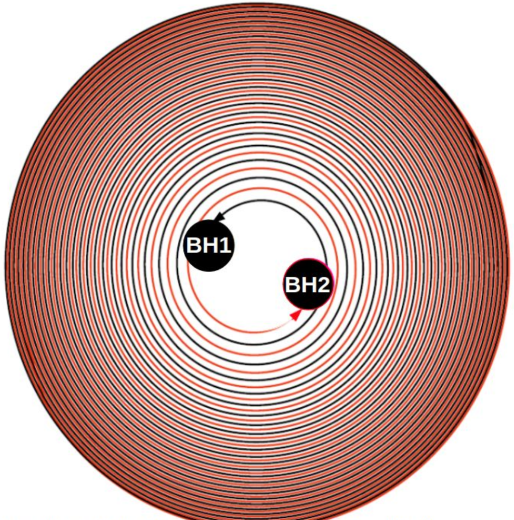

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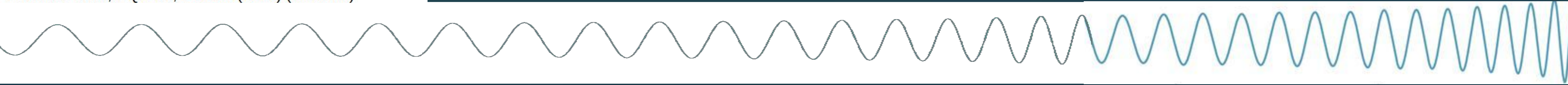

Time axis ⇨
(spans ~200ms)
Wave amplitude ⇧
(wave strain, arb. units)

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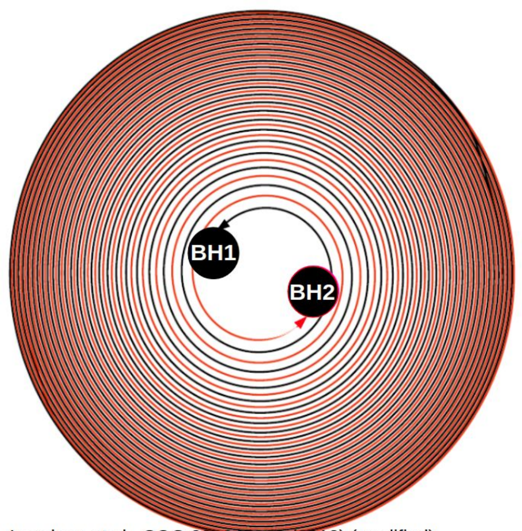

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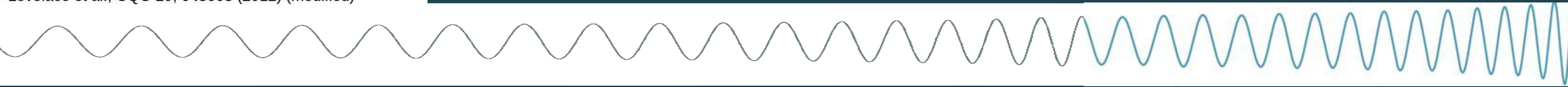

These waves encode info about masses, spins, and eccentricity of orbiting black holes

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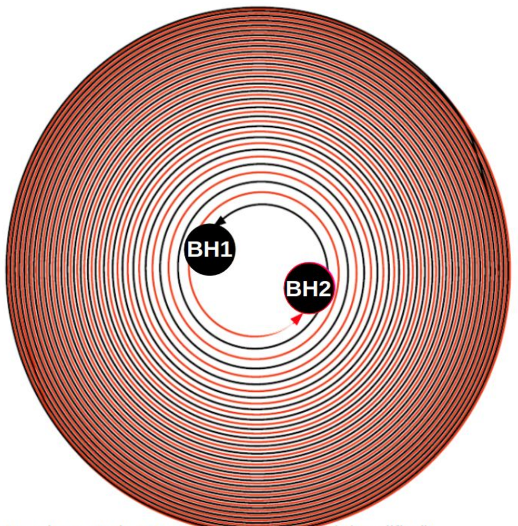

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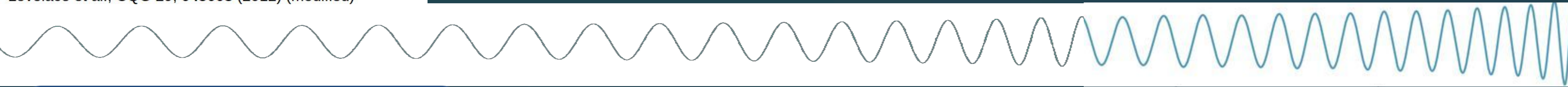

⇦ (Very) early inspiral: Perturbative solutions to Einstein gravity (GR)

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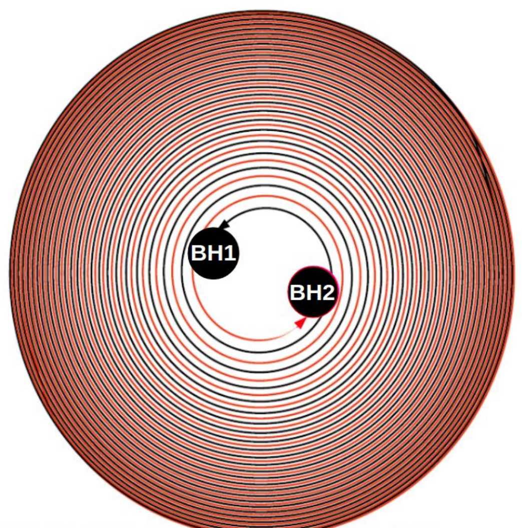

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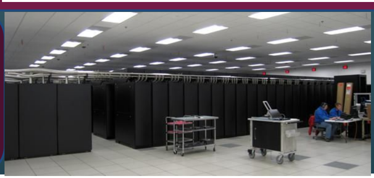
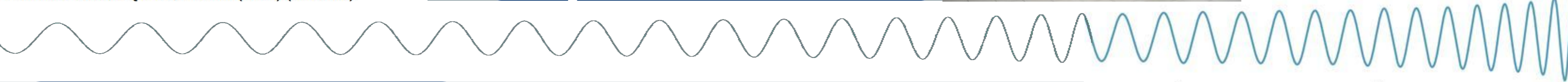
Late inspiral: Perturb. theory breaks down; **Only full GR solutions**





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Reformulate Einstein's theory of gravity for the computer

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Most popular formulation

NRPy+: Automated code generation for numerical relativity

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ICERM NRPy+ tutorial
Thursday, Oct 15; time TBD

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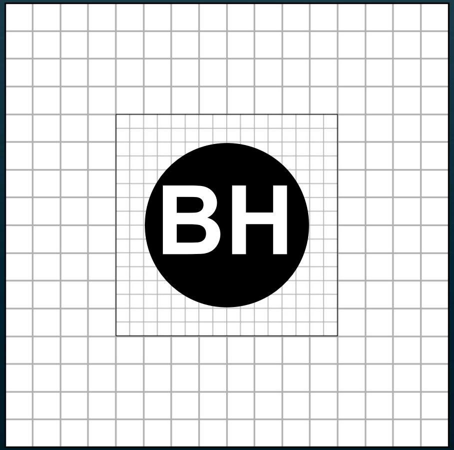

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How to address?

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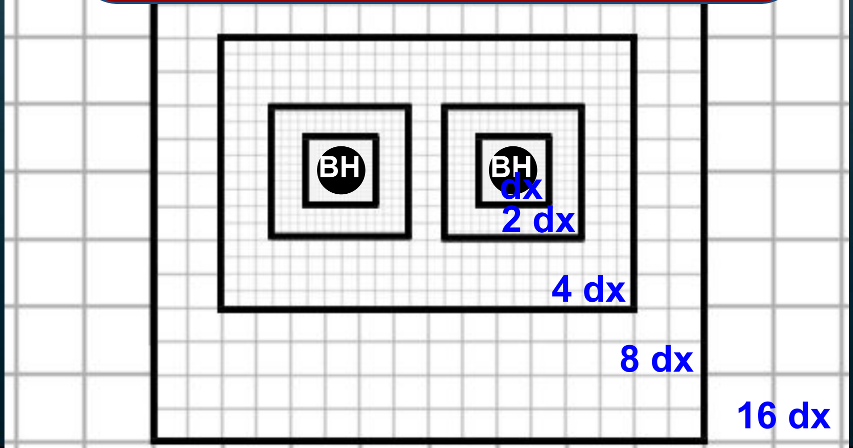
AMR

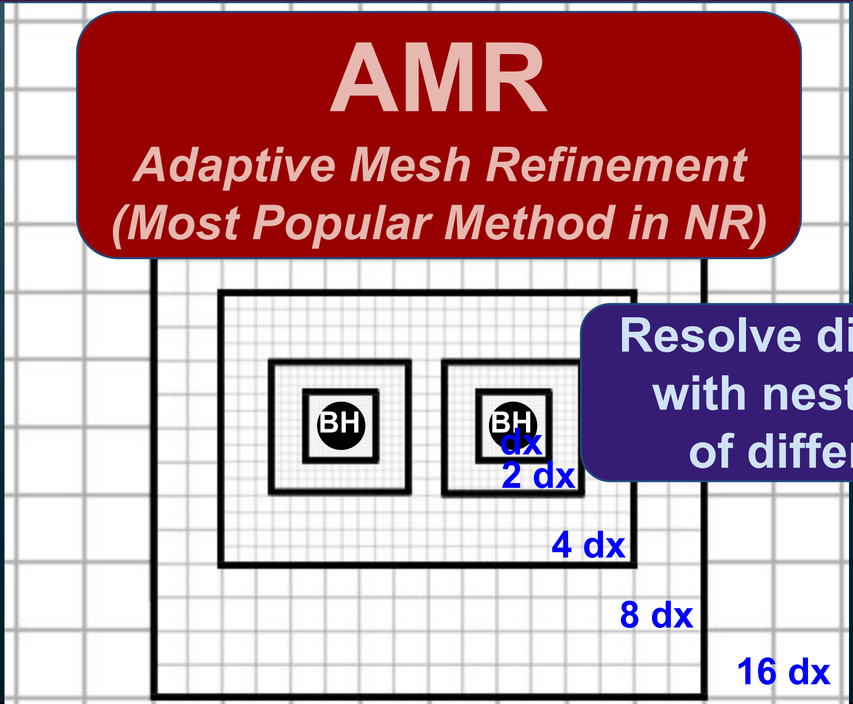
Adaptive Mesh Refinement
(Most Popular Method in NR)



Modeling Challenges

Address (~5 orders of mag) disparity in physical scales

1. Resolve **sharp**, rapidly changing grav fields near BHs and NSs
2. Model **long-wavelength** gravitational waves far away
3. Push outer boundary very far away (due to approx. BCs)



AMR

Adaptive Mesh Refinement
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Resolve disparate lengthscales with nested Cartesian cubes of differing grid spacings

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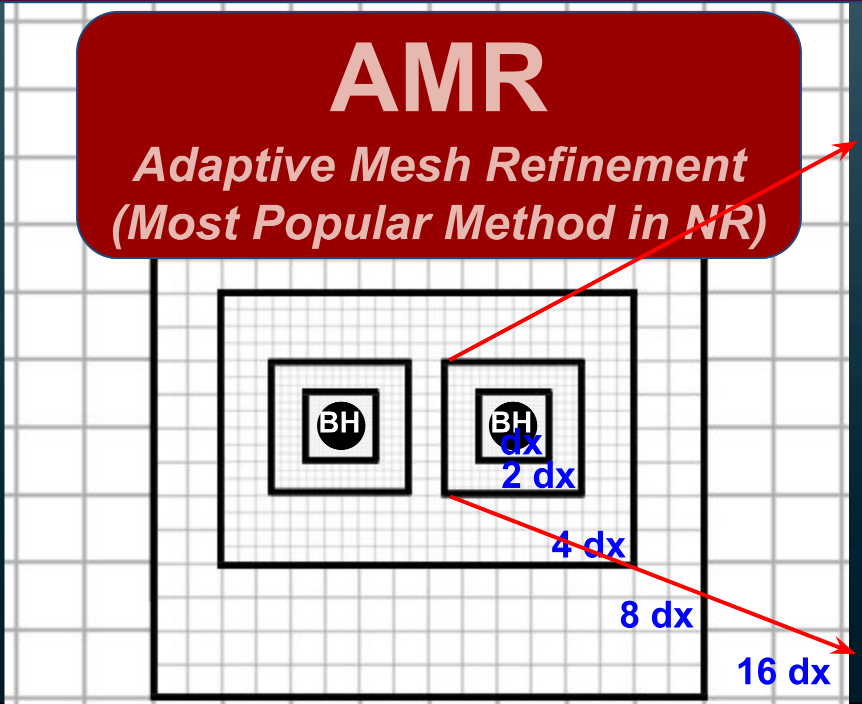

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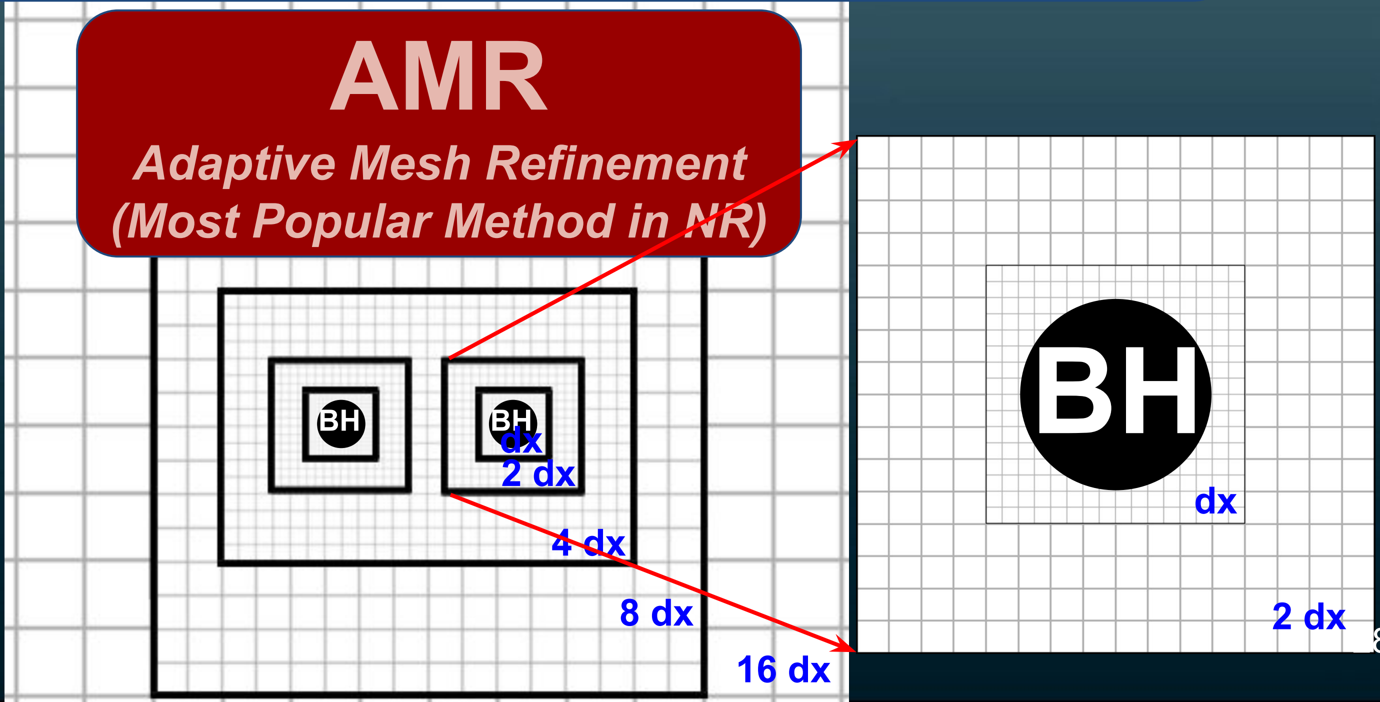

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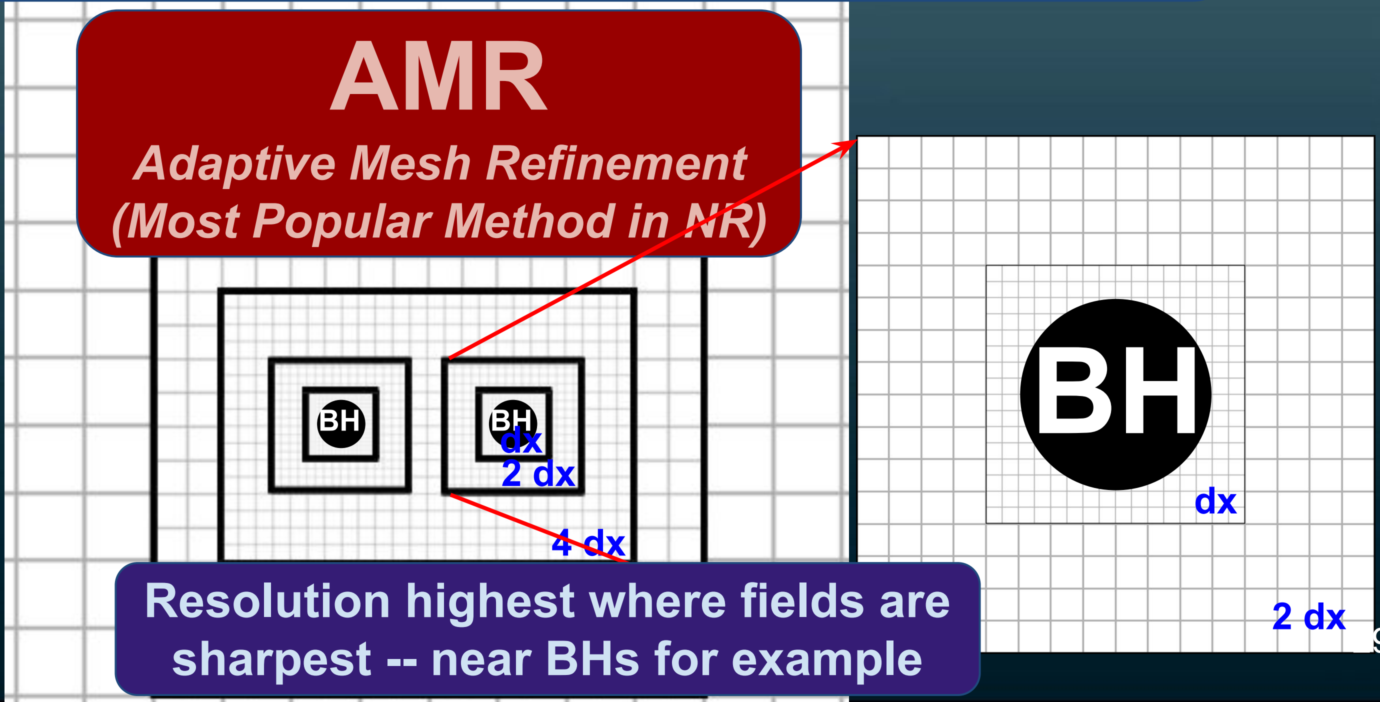

Resolution highest where fields are sharpest -- near BHs for example

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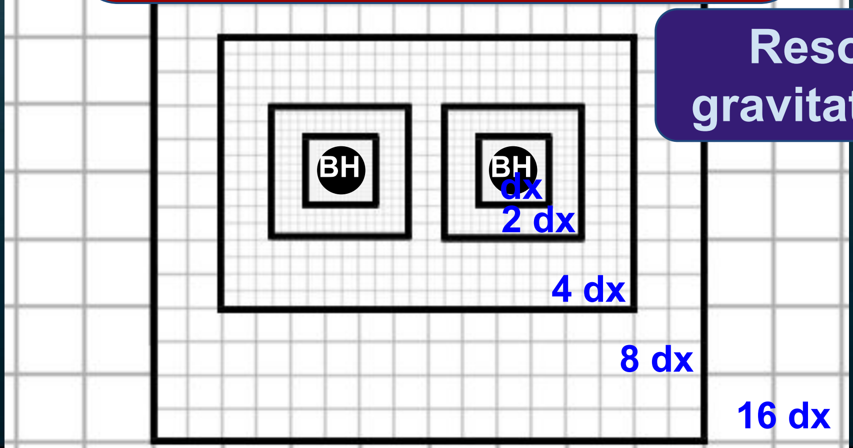

Resolution lower where gravitational waves modeled

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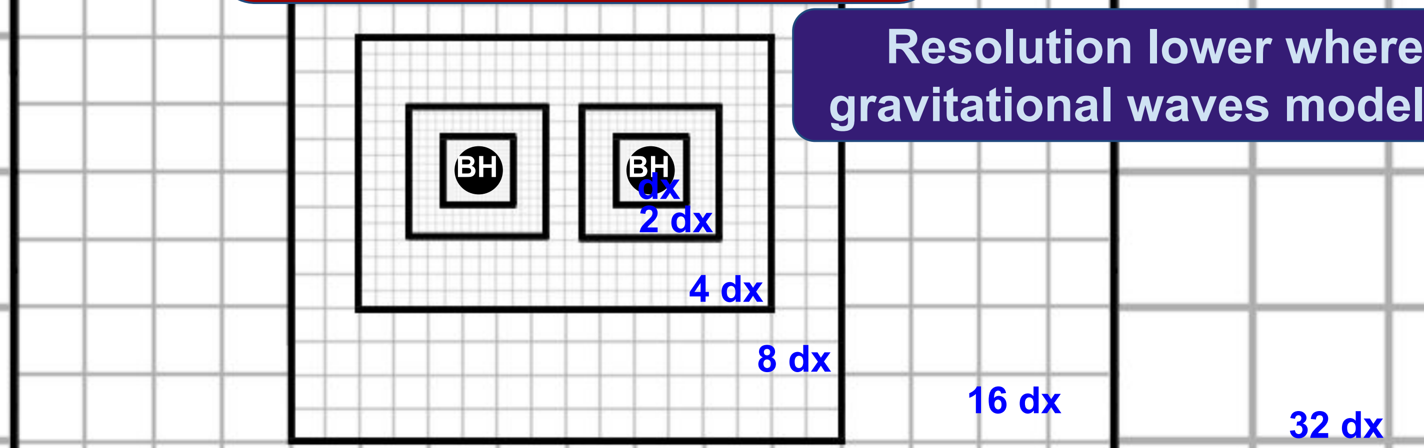

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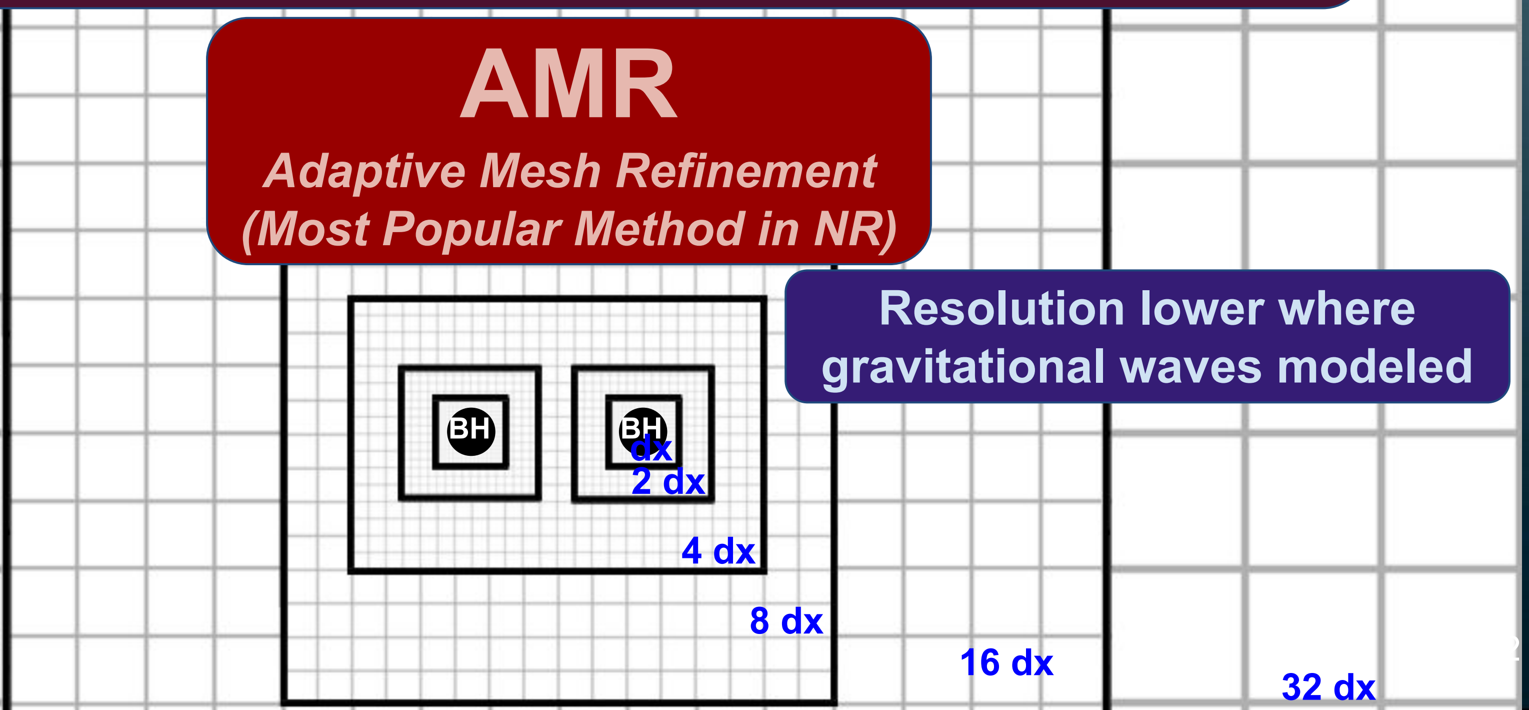

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AMR

Adaptive Mesh Refinement
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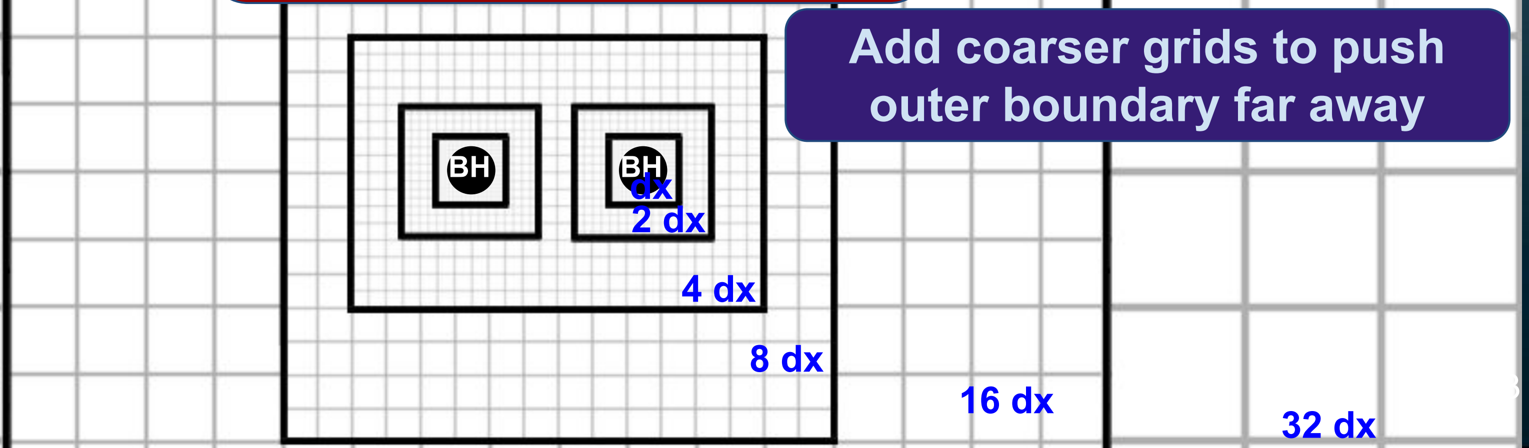
Add coarser grids to push outer boundary far away



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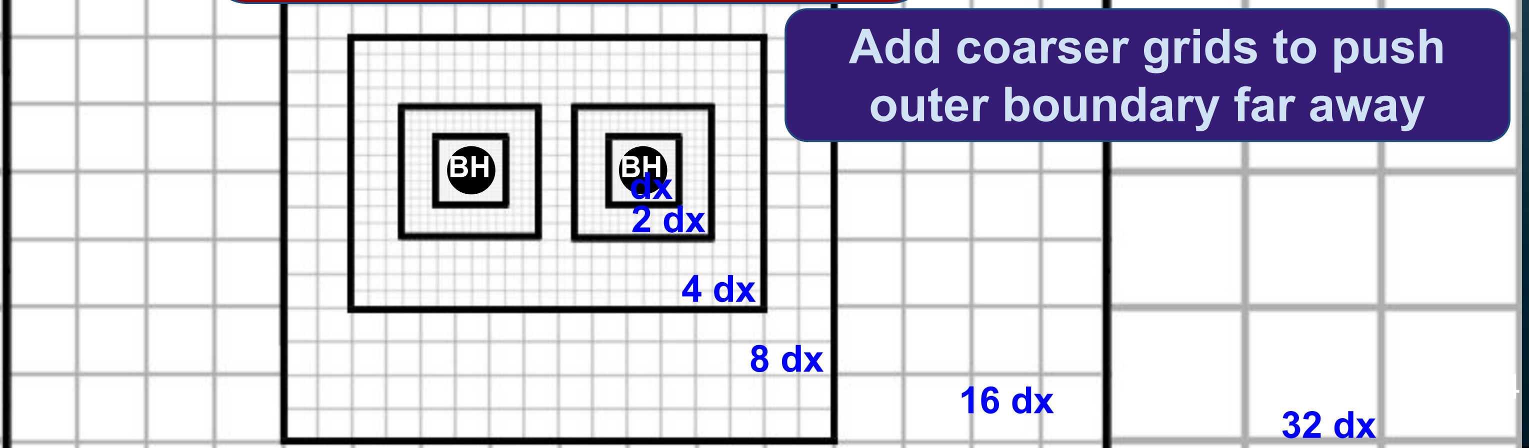

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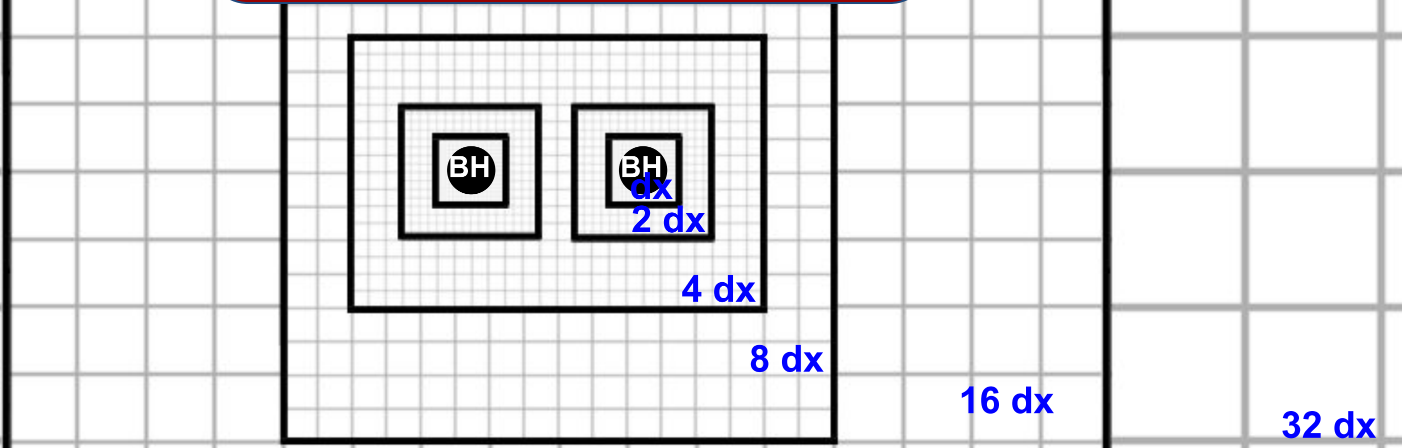

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What's the problem?

AMR

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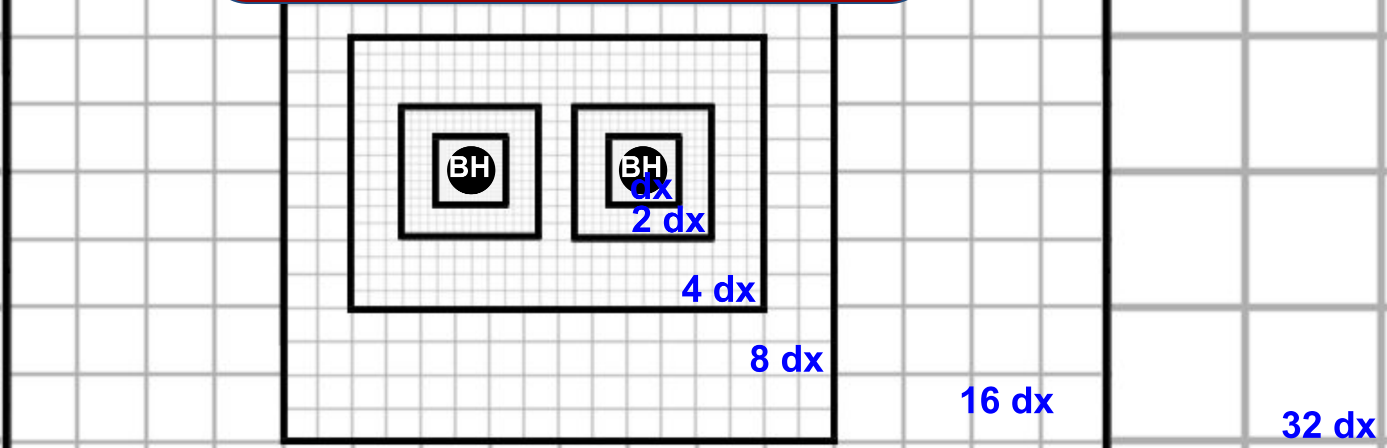

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What's the problem?

A: Inefficient!
⇨ greater comp. cost

AMR

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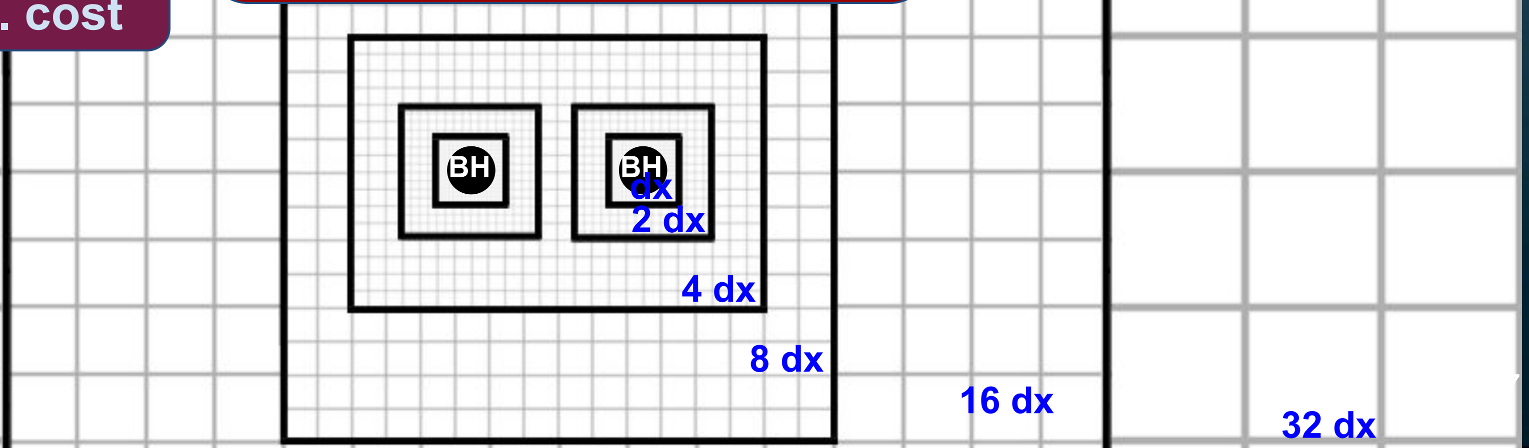

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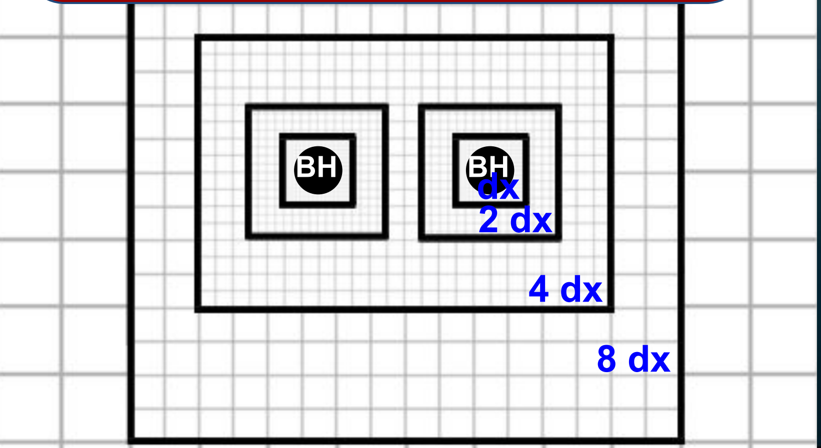
AMR Grid *Inefficiencies*



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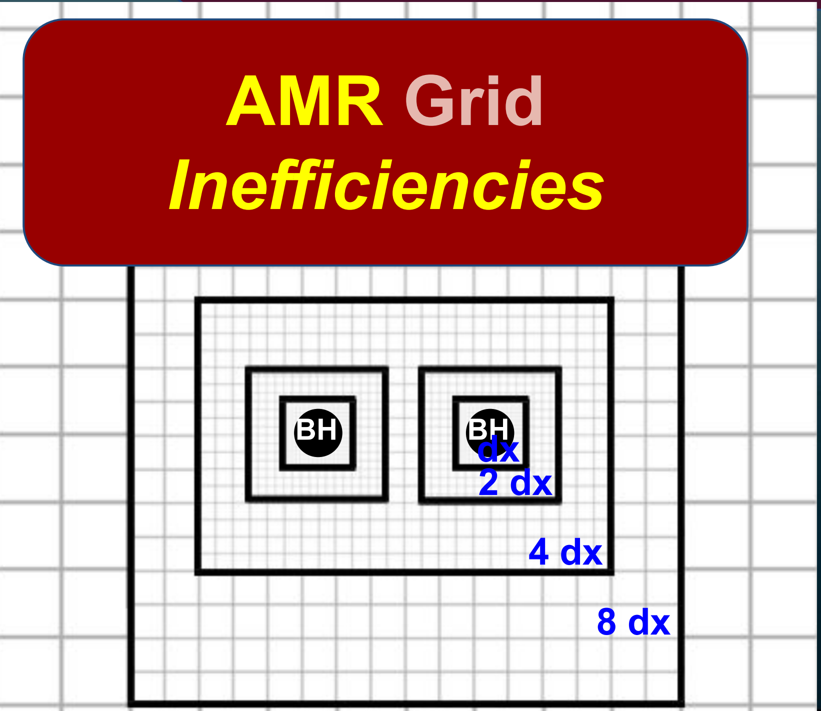

1. Black holes & gravitational waves: nearly spherical/axisymmetric
   - ⇨ grav fields vary most strongly in *radial* direction
     ⇨ <u>need high sampling in *only one* (*r*) direction</u>

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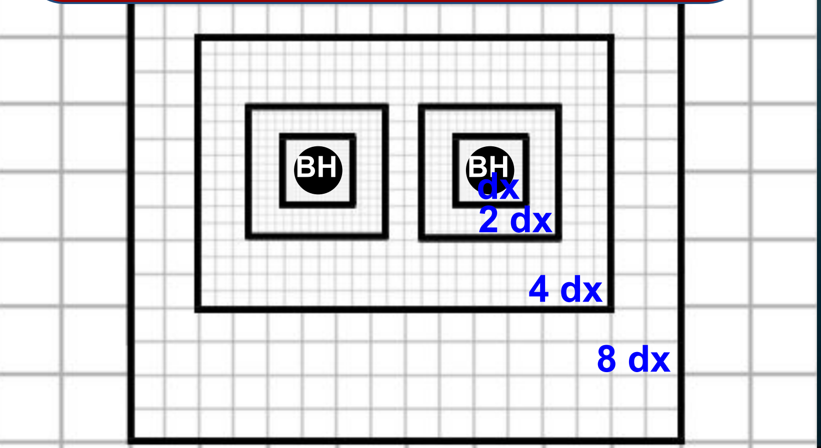

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   - ⇨ grav fields vary most strongly in *radial* direction
     ⇨ <u>need high sampling in *only one* (*r*) direction</u>

2. <u>Gravitational fields are mostly smooth</u>
   - Cartesian **AMR** grids:
     - 2x jumps in resolution between boxes
     - Boxes have sharp corners

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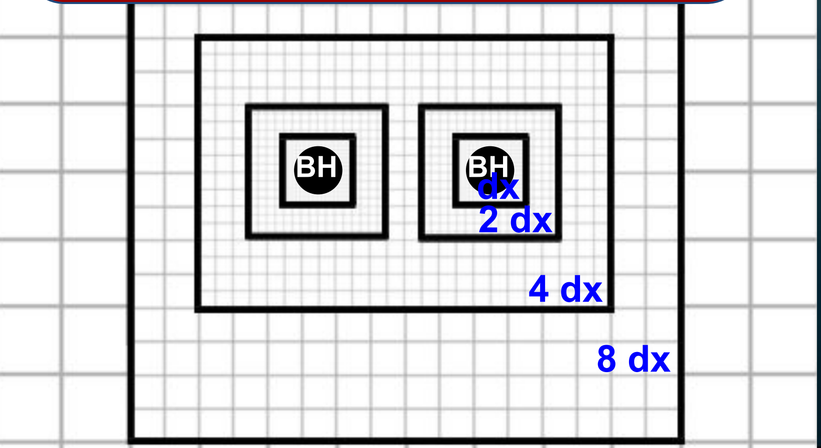
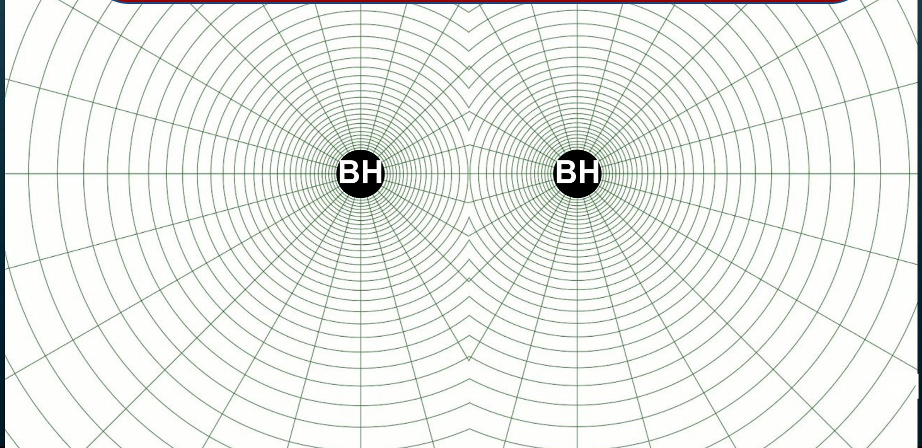


New BiSphere Grids

~20x more efficient sampling for compact binary simulations



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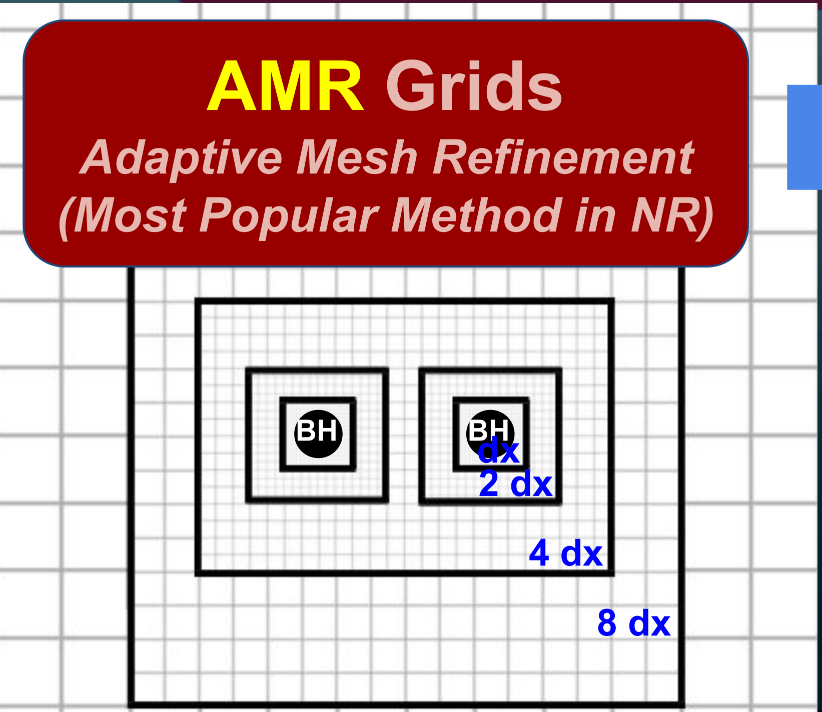

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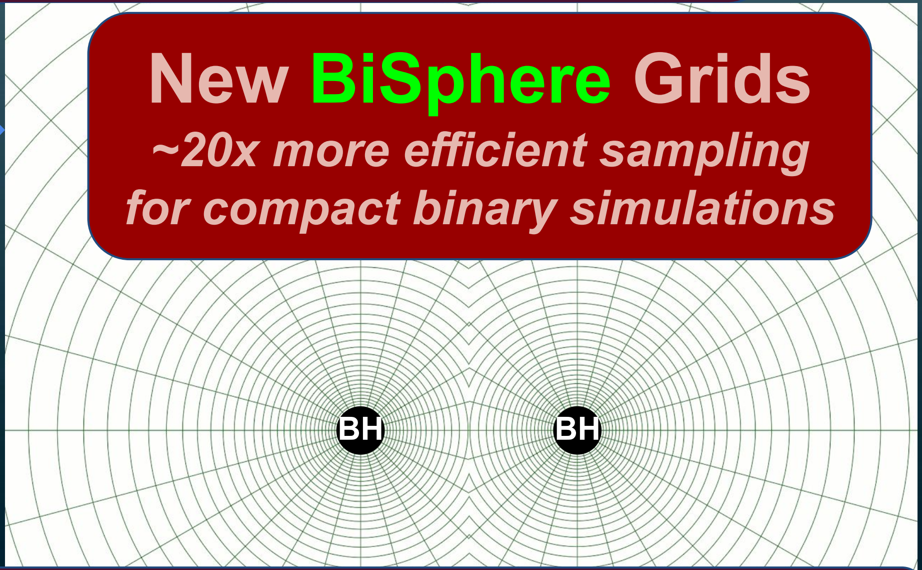
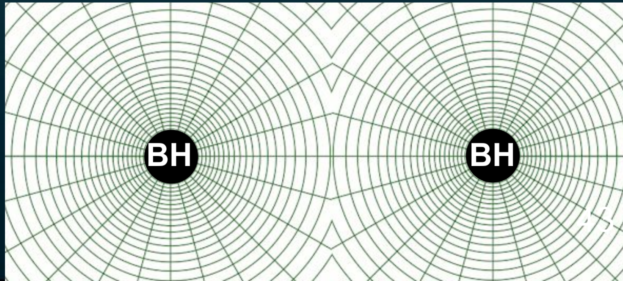


- **Exploits near-symmetries (~5x)**
- **Smooth transitions in resolution (~4x)**

BiSphere Challenges

BiSphere grids: Two overlapping numerical grids in Spherical coordinates



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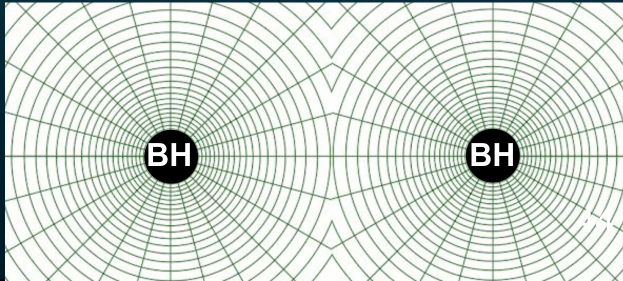
  - Tensors and vectors diverge or go to zero -> numerically **unstable**



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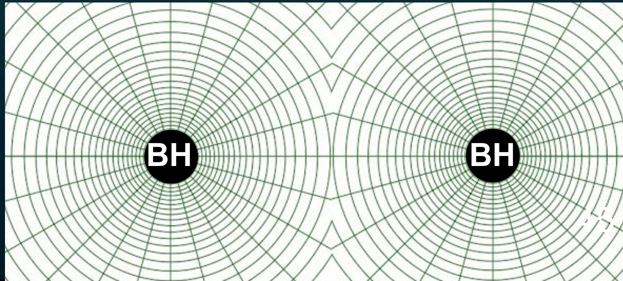

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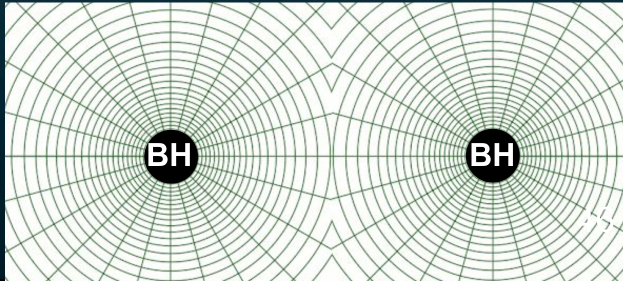
a. **Ordinary spherical polar: done!**

Baumgarte, Montero, Cordero-Carrión, Müller (PRD 87, 044026, **2012**),
built upon covariant BSSN formulation of Brown (PRD 79, 104029, **2009**)

b. **Generic-radius spherical polar (incl. log-radial): done!**

Ruchlin, **Etienne**, Baumgarte (PRD 97, 064036, **2018**)



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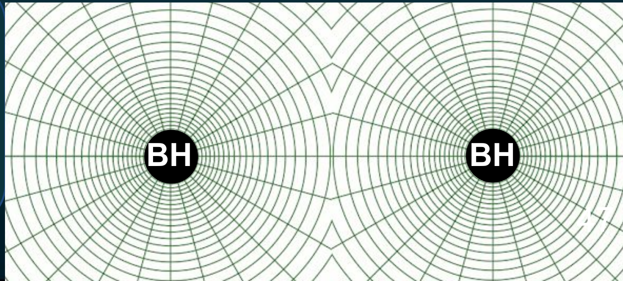

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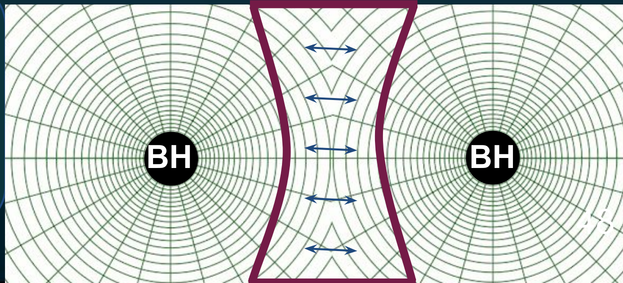
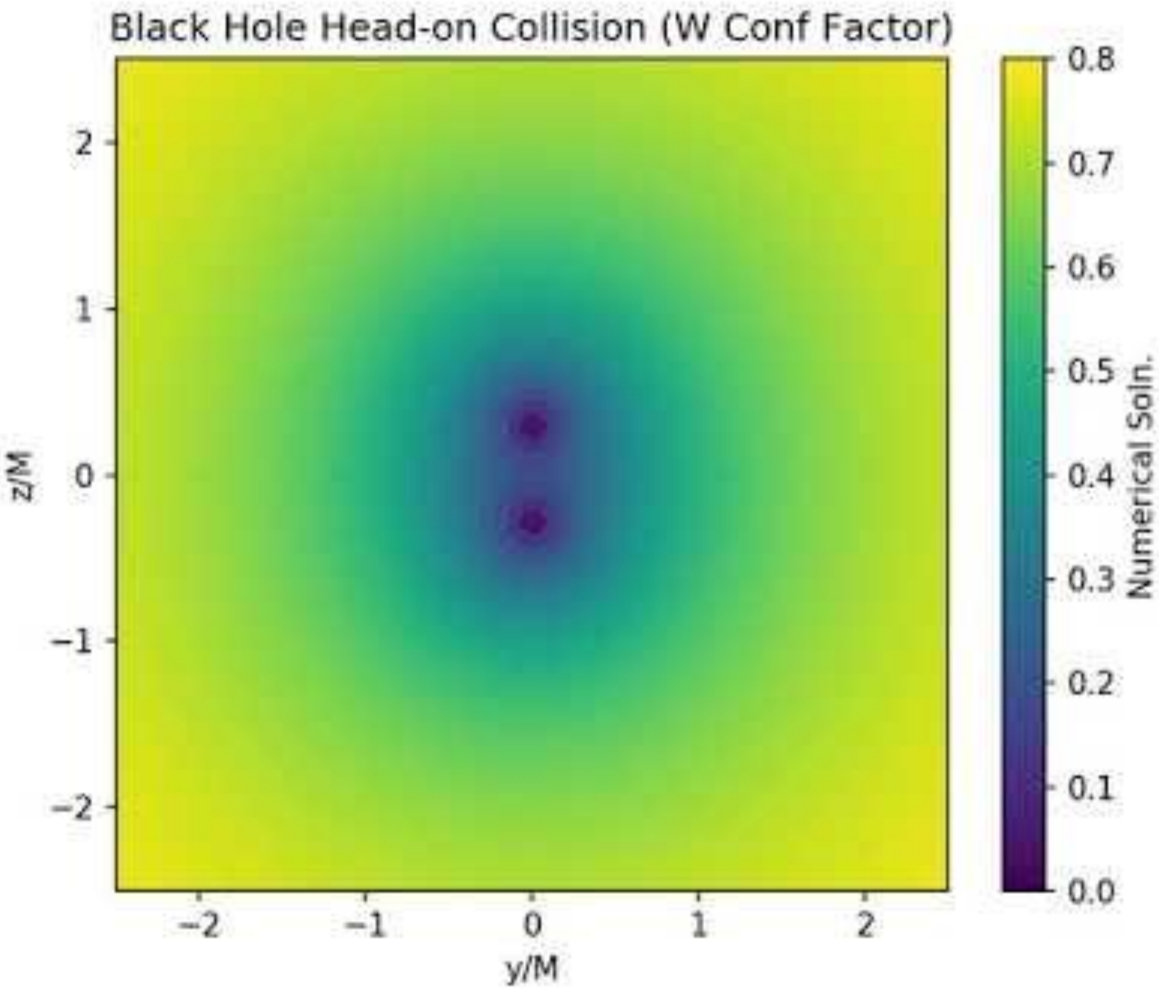


2019 Results



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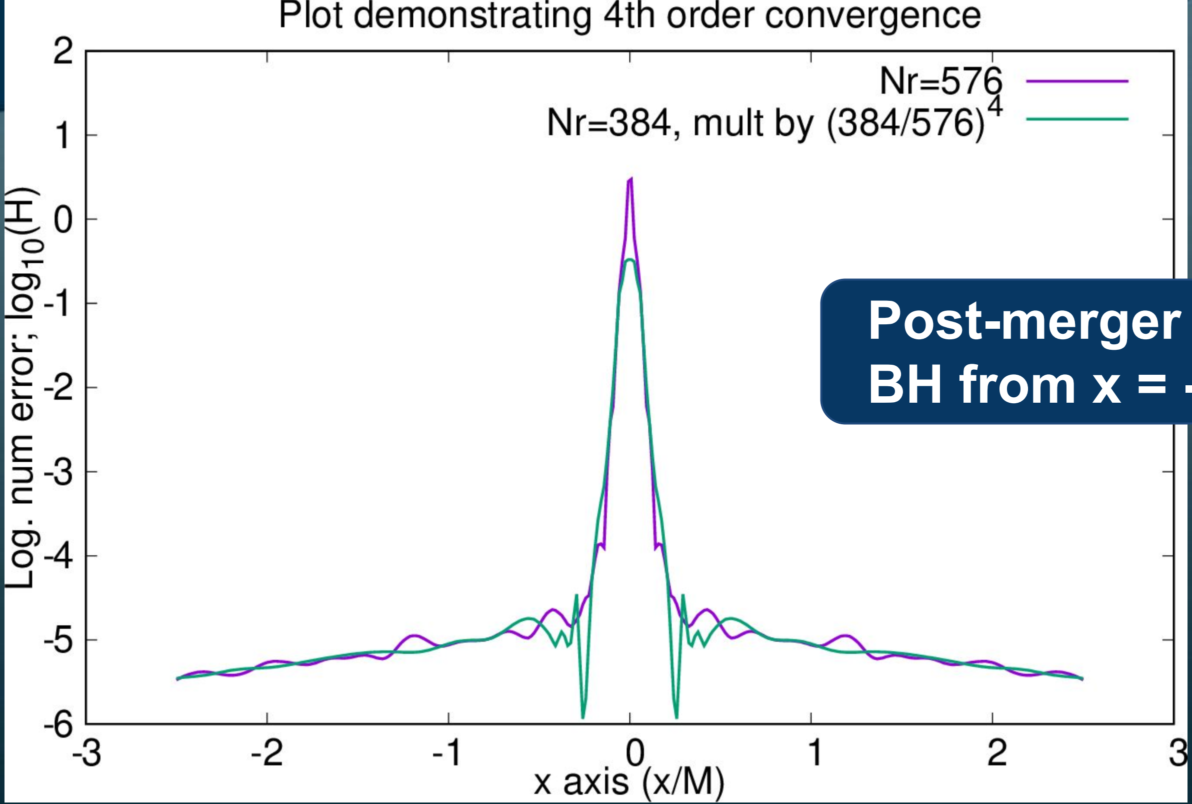

Post-merger num error, BH from x = -0.5 to +0.5

Finding from BH collision test: Numerical errors small and converge to zero at expected rate

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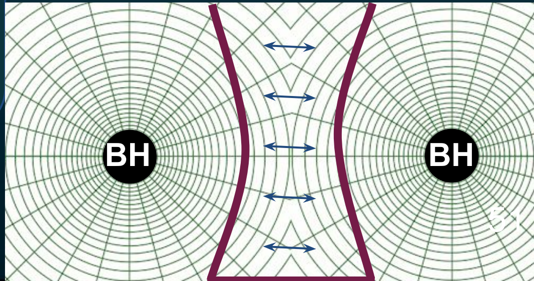

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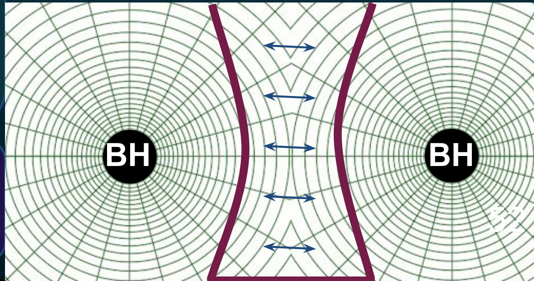
- Challenge #3: Calculations *50x* too slow: *small timesteps!*



Small Timesteps,
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- Problem:
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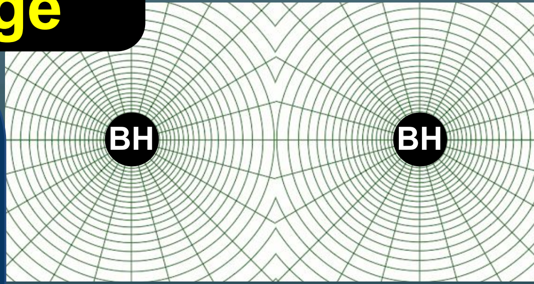

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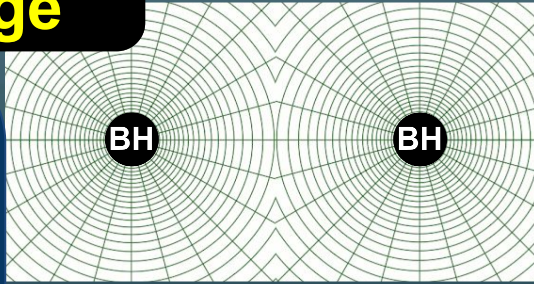

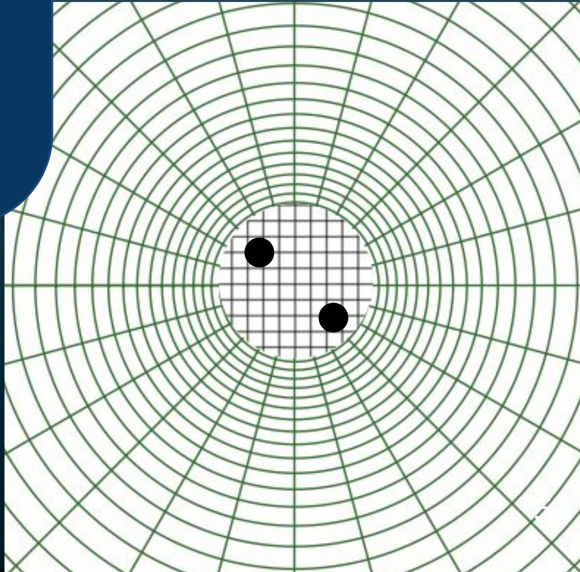

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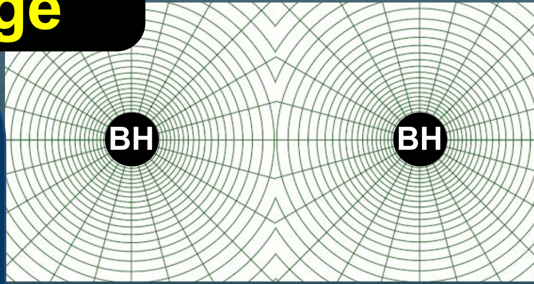
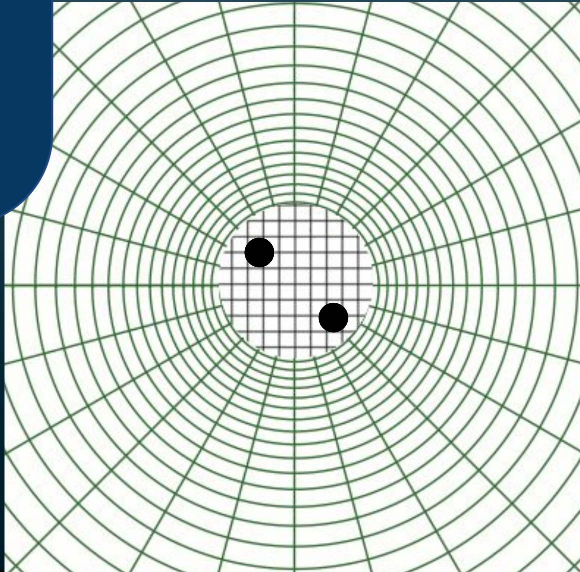
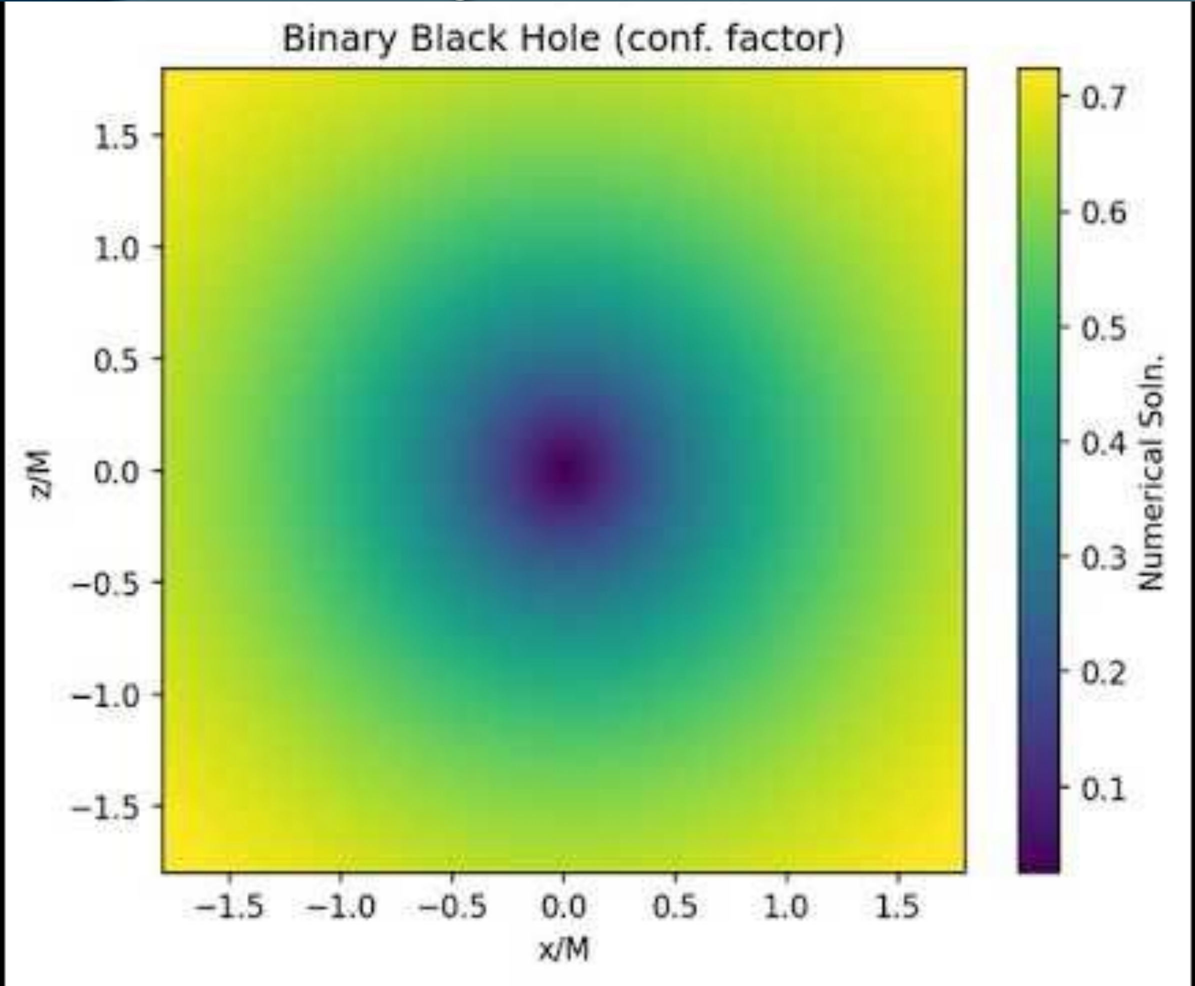




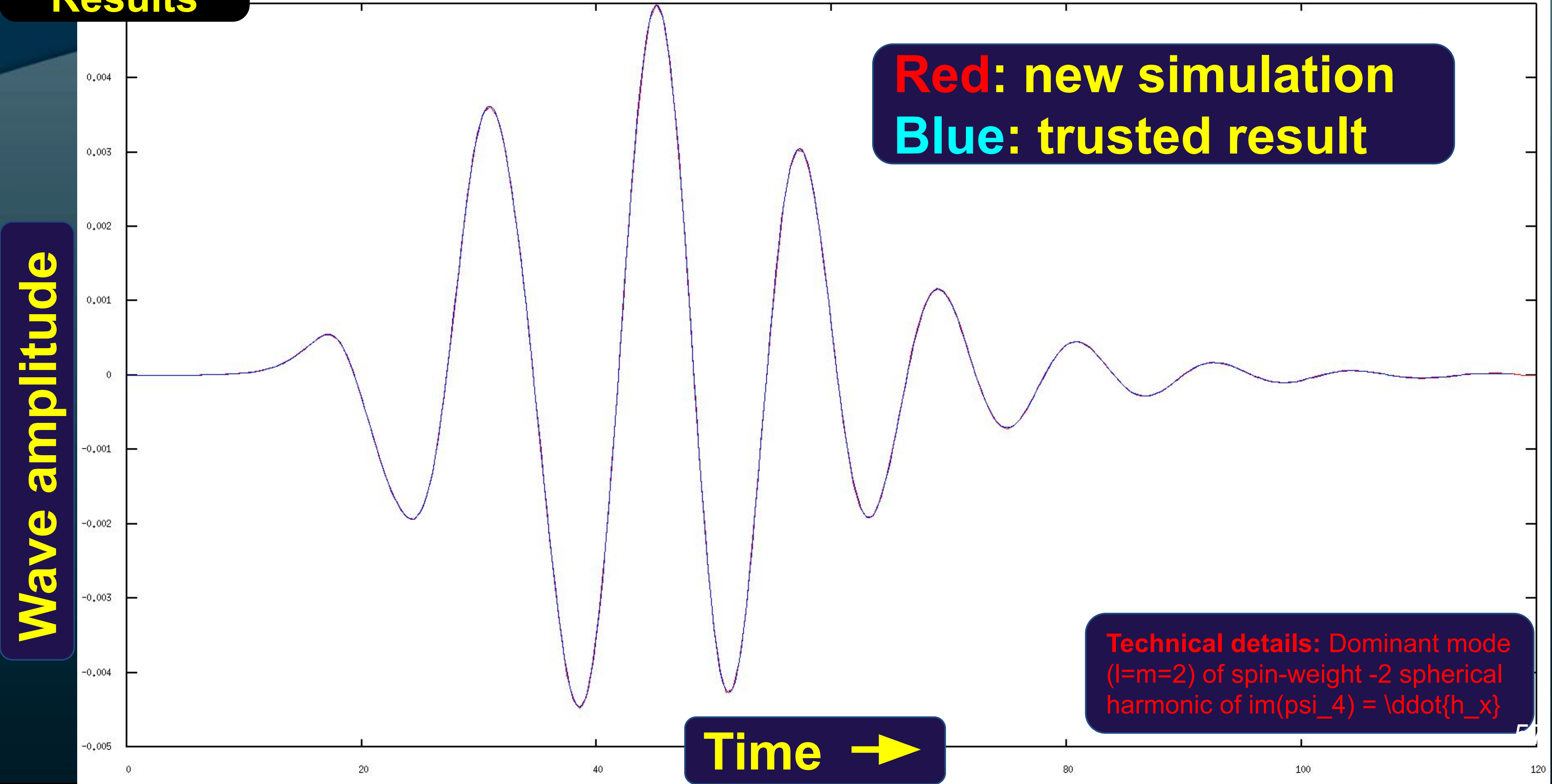
Will it work?!

Early 2020 Results



Early 2020 Results

Gravitational Wave Comparison



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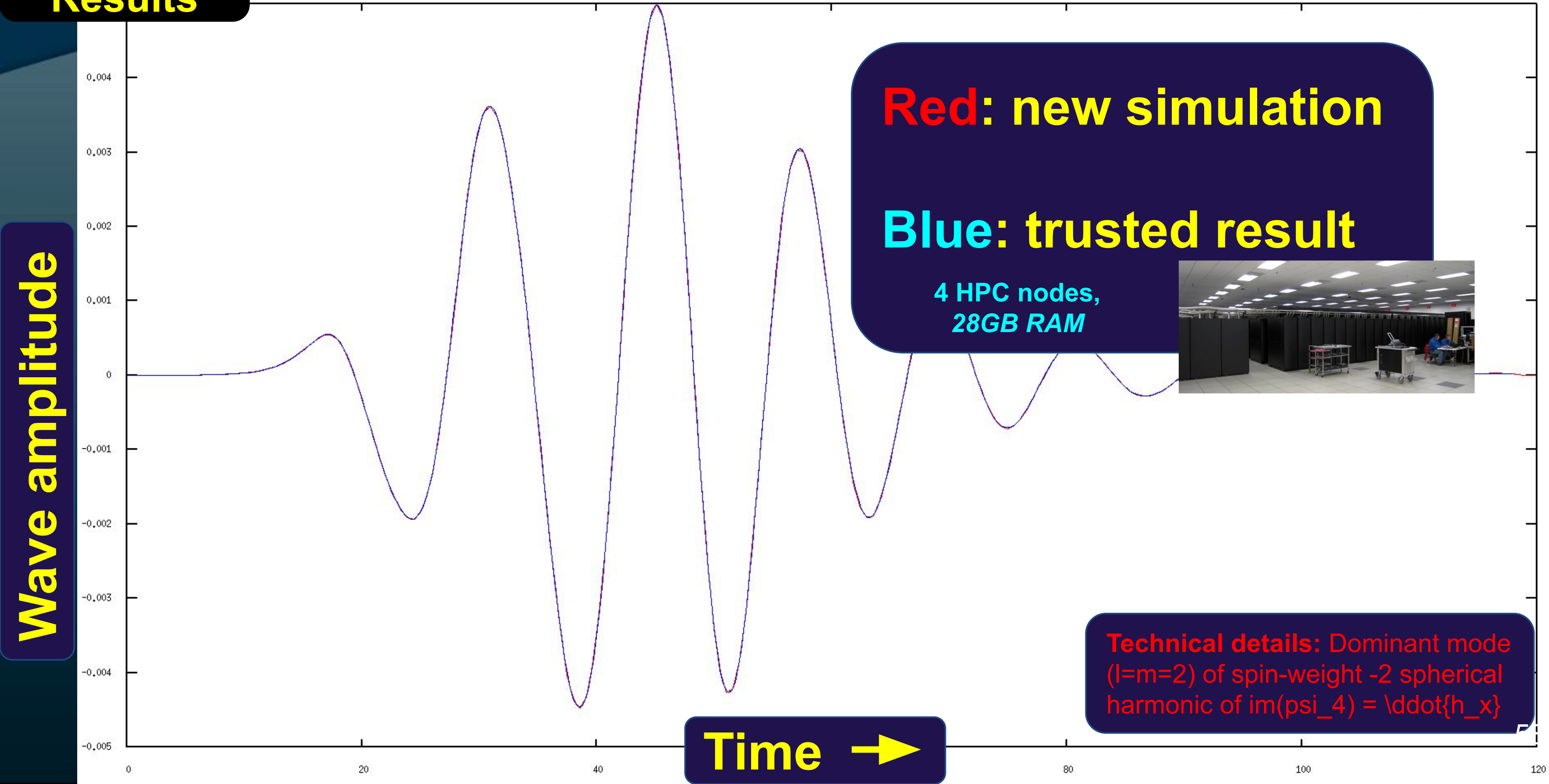

Red: new simulation

Blue: trusted result

4 HPC nodes,
28GB RAM

Technical details: Dominant mode (l=m=2) of spin-weight -2 spherical harmonic of im(psi_4) = \ddot{h_x}

Early 2020 Results

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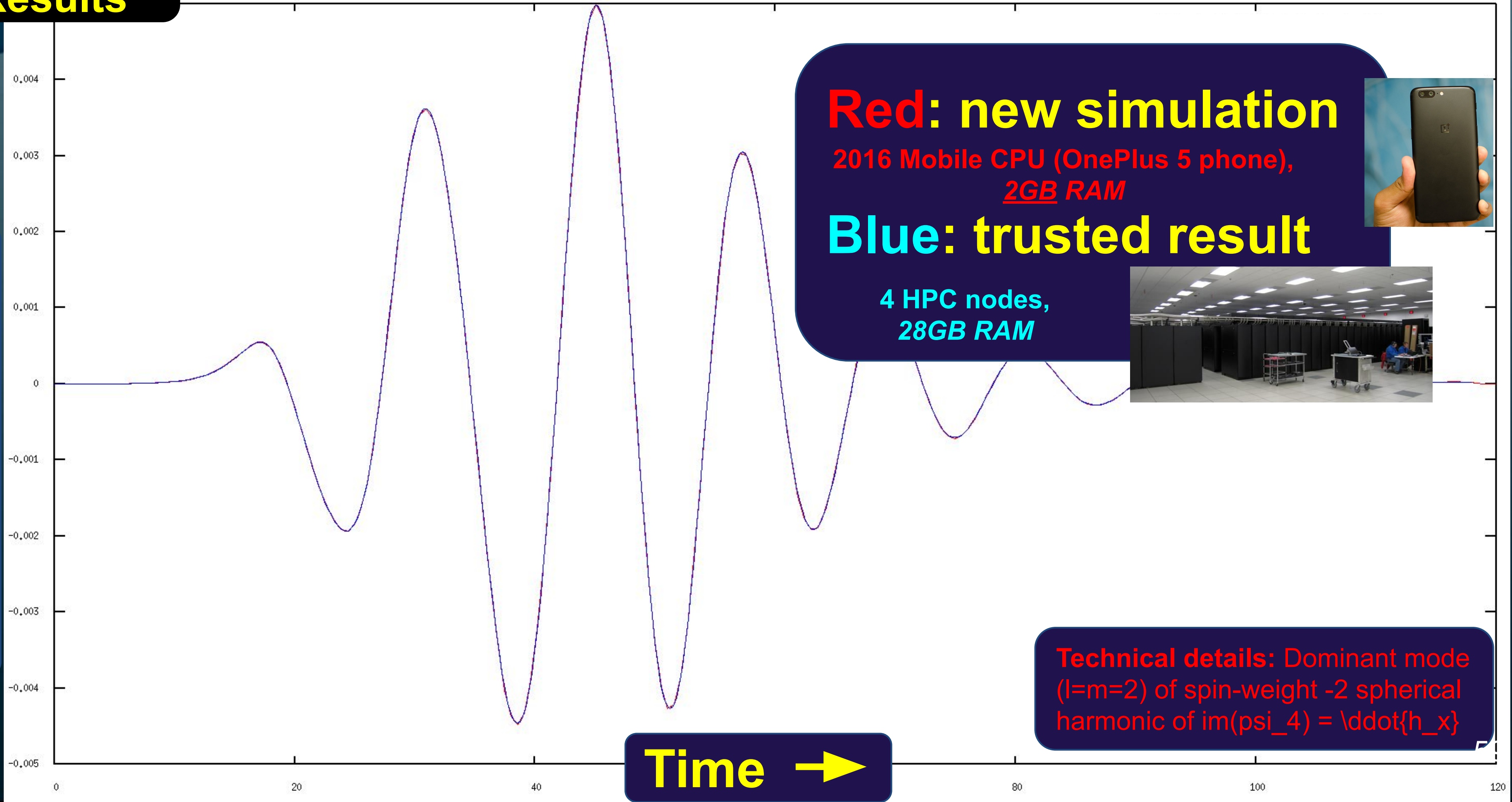

Wave amplitude

Time →

Red: new simulation

2016 Mobile CPU (OnePlus 5 phone), *2GB RAM*

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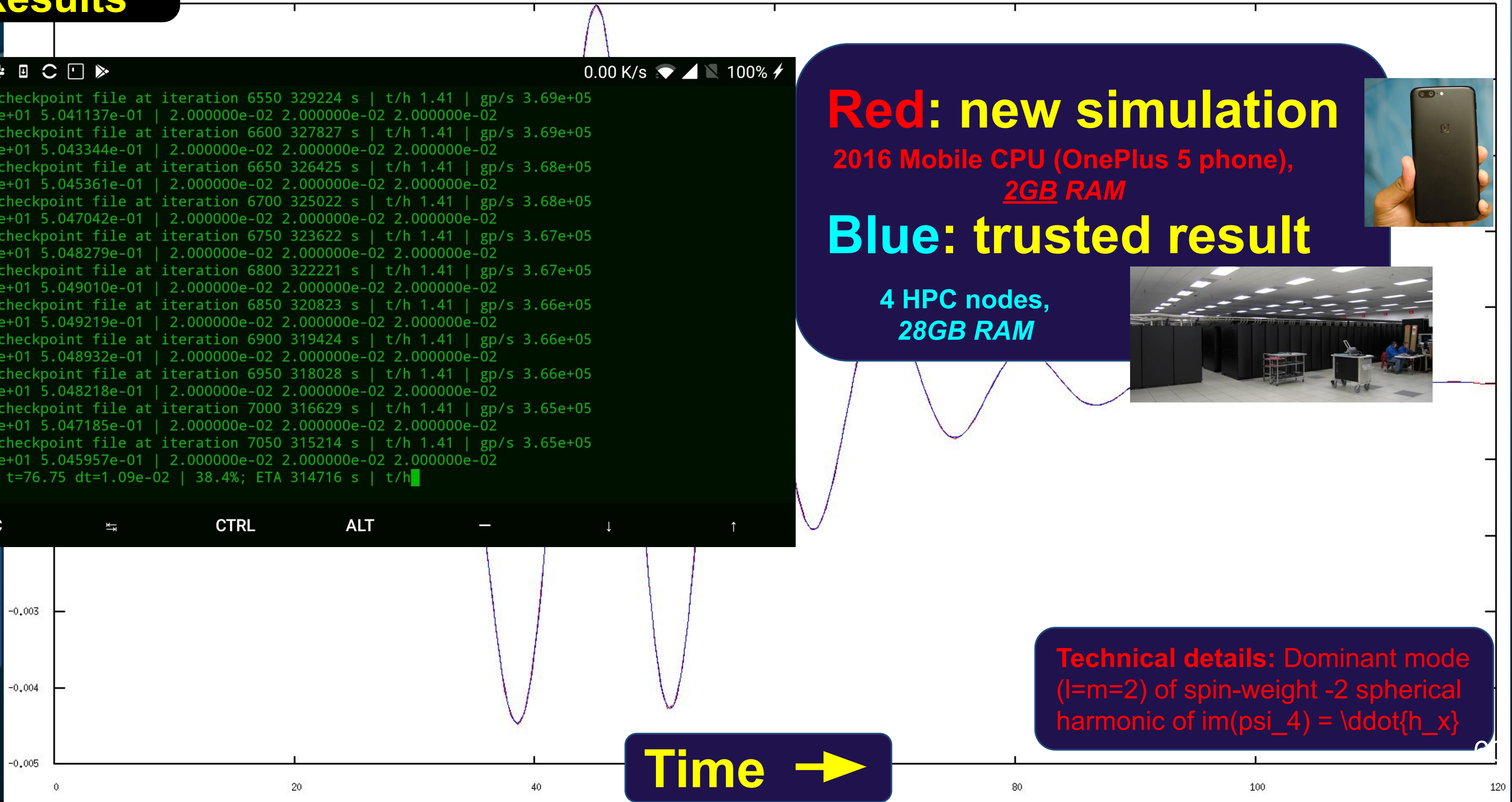
Early 2020 Results

Gravitational Wave Comparison

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16:29                                   0.00 K/s   100%
Writing checkpoint file at iteration 6550 329224 s | t/h 1.41 | gp/s 3.69e+05
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Writing checkpoint file at iteration 6650 326425 s | t/h 1.41 | gp/s 3.68e+05
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Writing checkpoint file at iteration 6700 325022 s | t/h 1.41 | gp/s 3.68e+05
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Writing checkpoint file at iteration 6750 323622 s | t/h 1.41 | gp/s 3.67e+05
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Writing checkpoint file at iteration 6950 318028 s | t/h 1.41 | gp/s 3.66e+05
7.548639e+01 5.048218e-01 | 2.000000e-02 2.000000e-02 2.000000e-02
Writing checkpoint file at iteration 7000 316629 s | t/h 1.41 | gp/s 3.65e+05
7.602946e+01 5.047185e-01 | 2.000000e-02 2.000000e-02 2.000000e-02
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It: 7066 t=76.75 dt=1.09e-02 | 38.4%; ETA 314716 s | t/h
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```

Red: new simulation

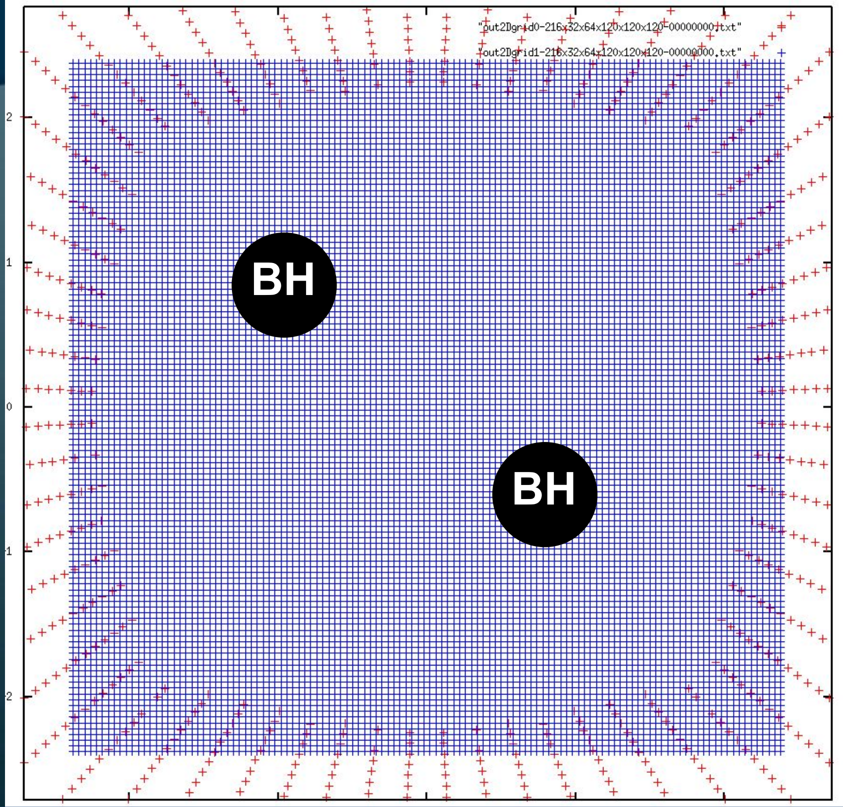
2016 Mobile CPU (OnePlus 5 phone), *2GB RAM*

Blue: trusted result

4 HPC nodes, *28GB RAM*

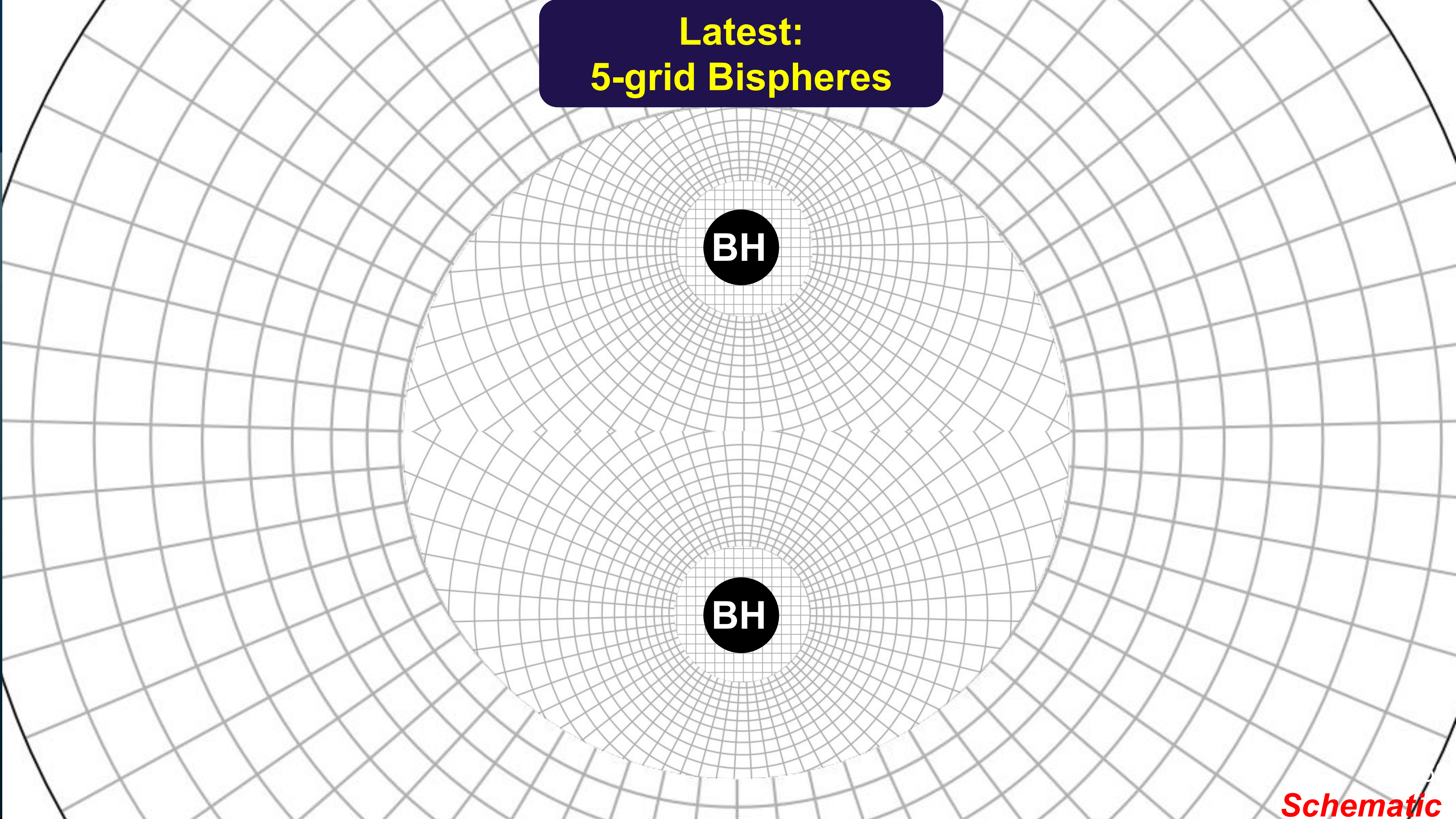
Technical details: Dominant mode (l=m=2) of spin-weight -2 spherical harmonic of im(psi_4) = \ddot{h_x}

Early 2020 Results



- Problem with this grid structure:
  - Only works well for two orbiting black holes ***very close* to merger**
  - Larger separations -- Cartesian grid too large -- too much memory!
  - Narrow Cartesian grid & rotate grids? Nope; resolution drop too large
- What to do?!

Latest:
5-grid Bispheres



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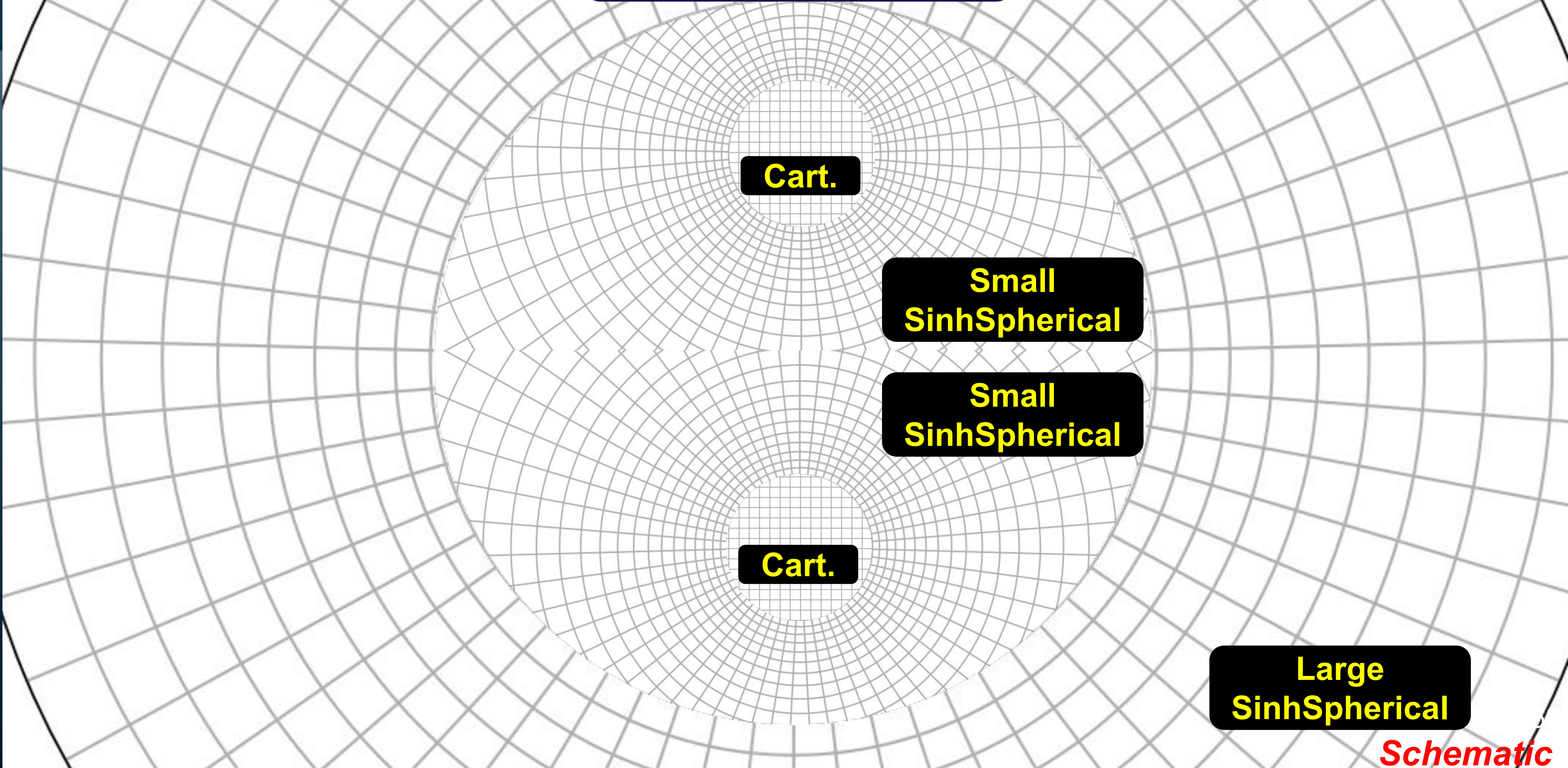
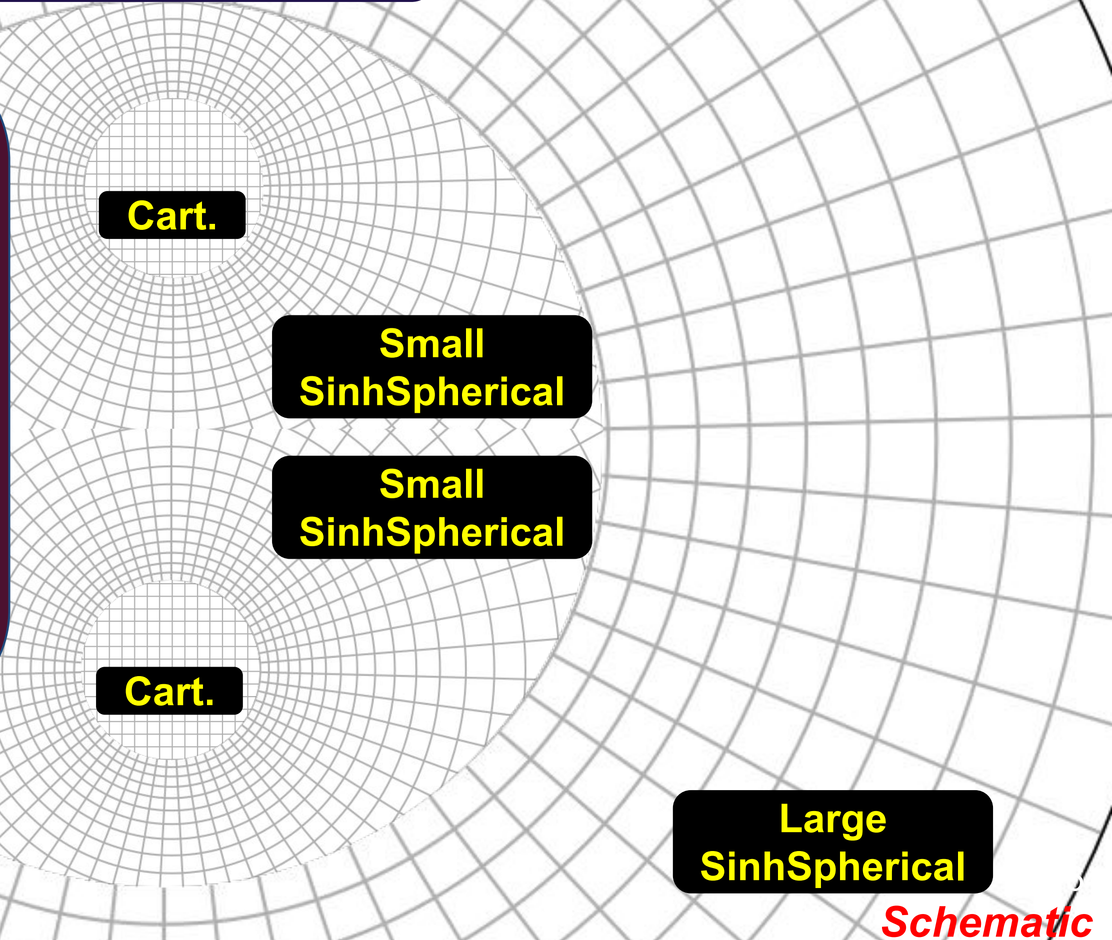


Latest: 5-grid Bispheres

Benefits

1. Cartesian grids maximize timestep where Sph grids would focus
2. *z*-axis collinear across all three SinhSpherical grids, Nphi fixed
   a. ⇒ 2D interpolations!
3. Large SinhSpherical *perfect* for GW extraction!
4. Low memory footprint
5. Intergrid surface area << Cartesian AMR ⇒ Could scale well!



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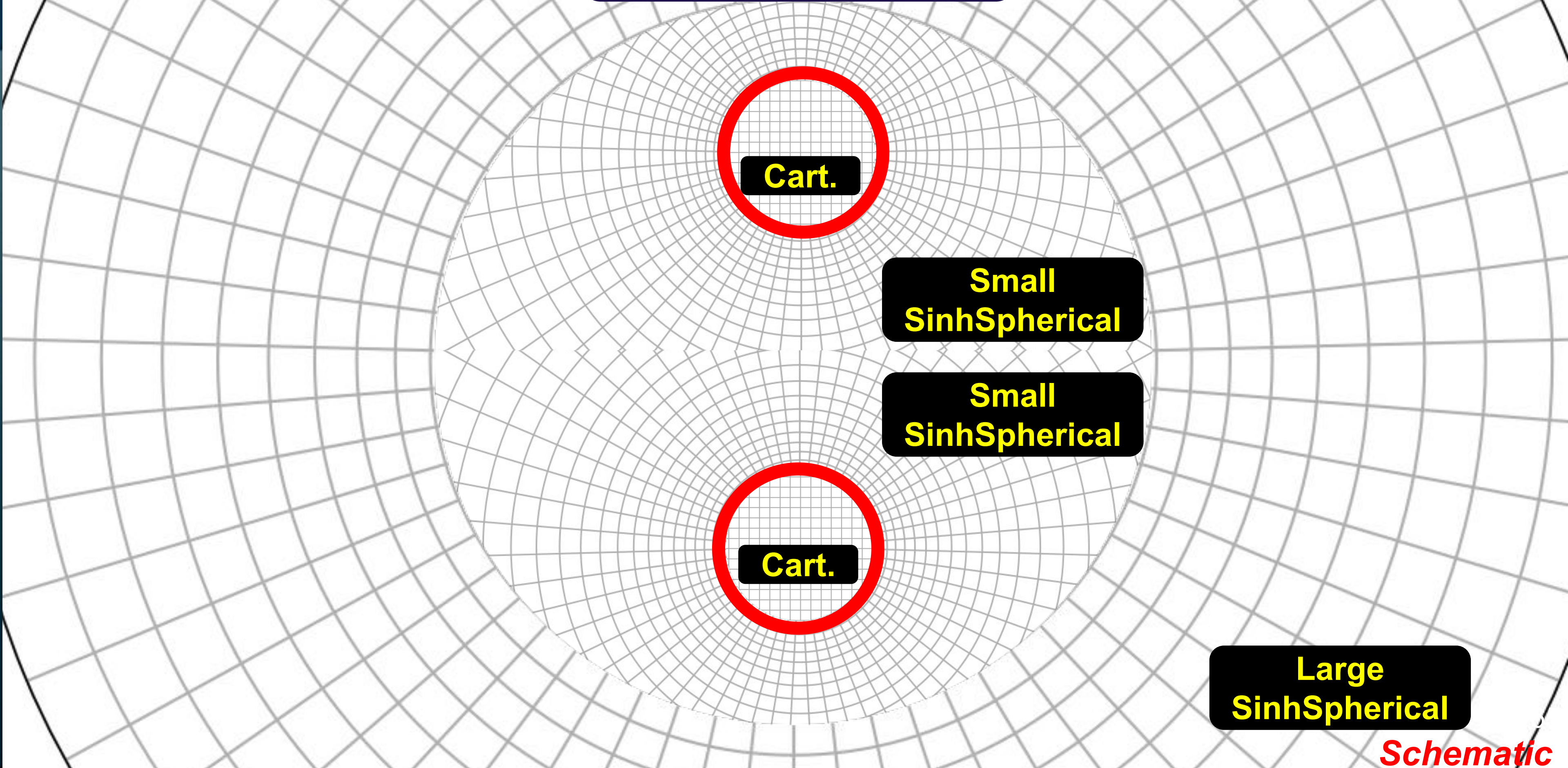
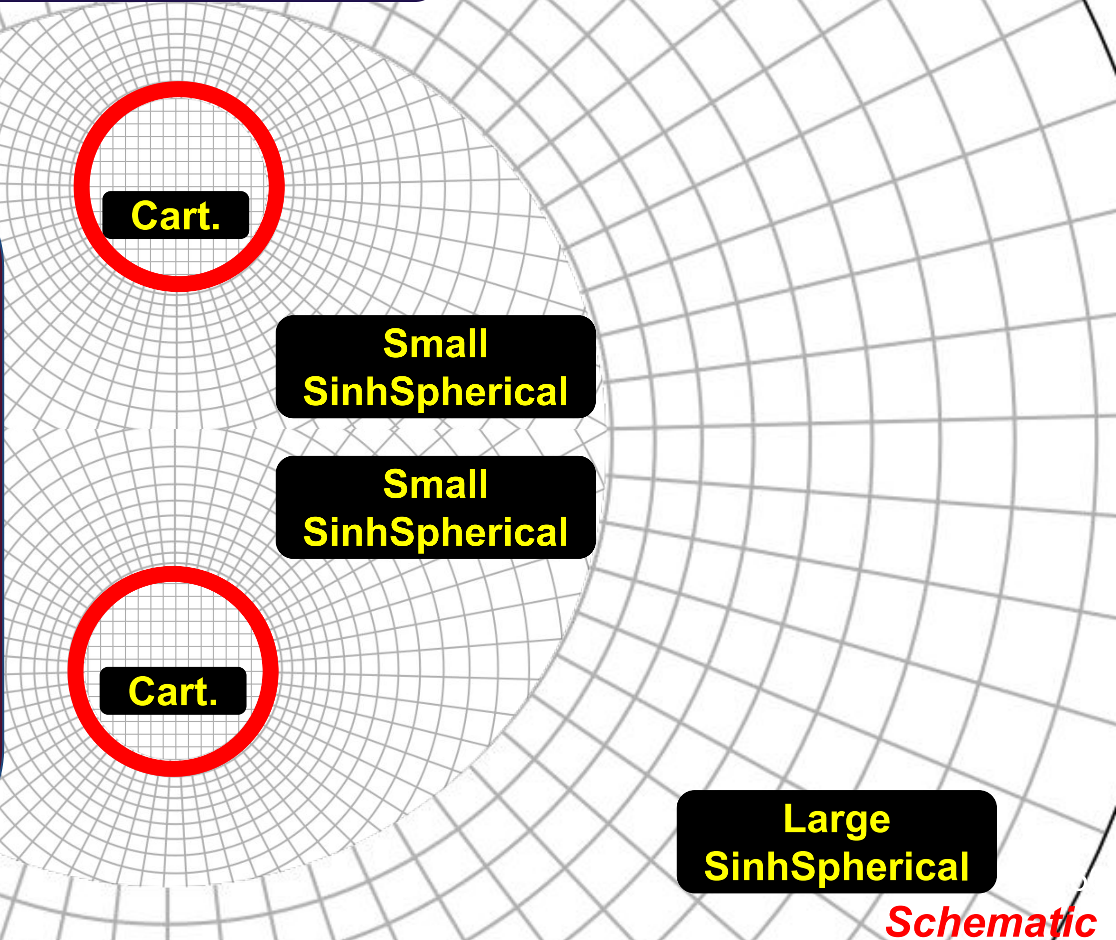


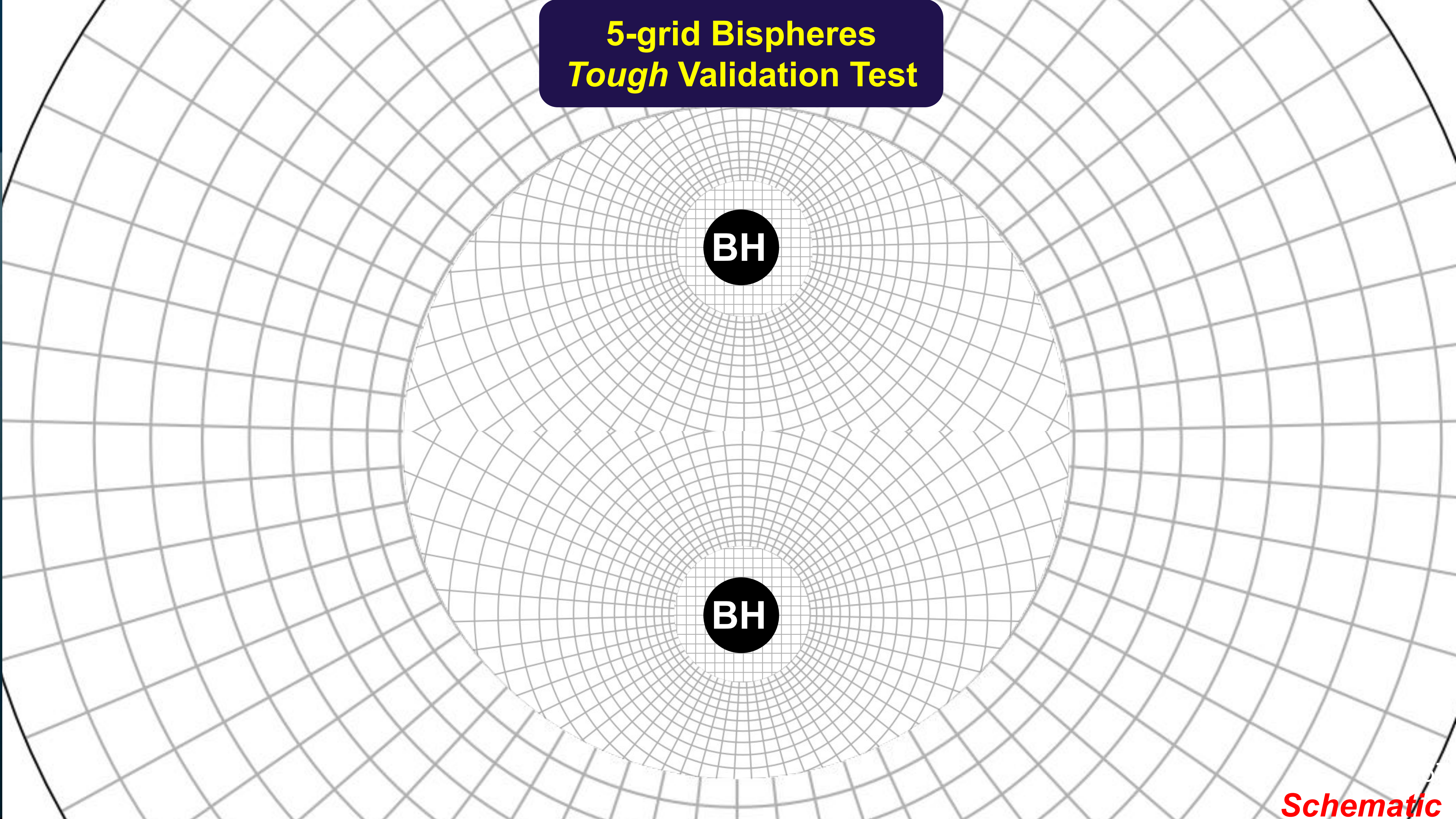
Latest:
5-grid Bispheres

<u>***Tough* Validation Test**</u>

- Fix grids in place (static grids)
- Match resolutions at red circles
  - (SinhSpherical/Cartesian intergrid boundary)
- Increase resolution near CoM
- Release BHs from rest, allowing BHs to cross intergrid boundaries
- BH merger remnant sits at intergrid boundary (SinhSpherical/SinhSpherical)
- Gravitational waves?!



5-grid Bispheres
Tough Validation Test



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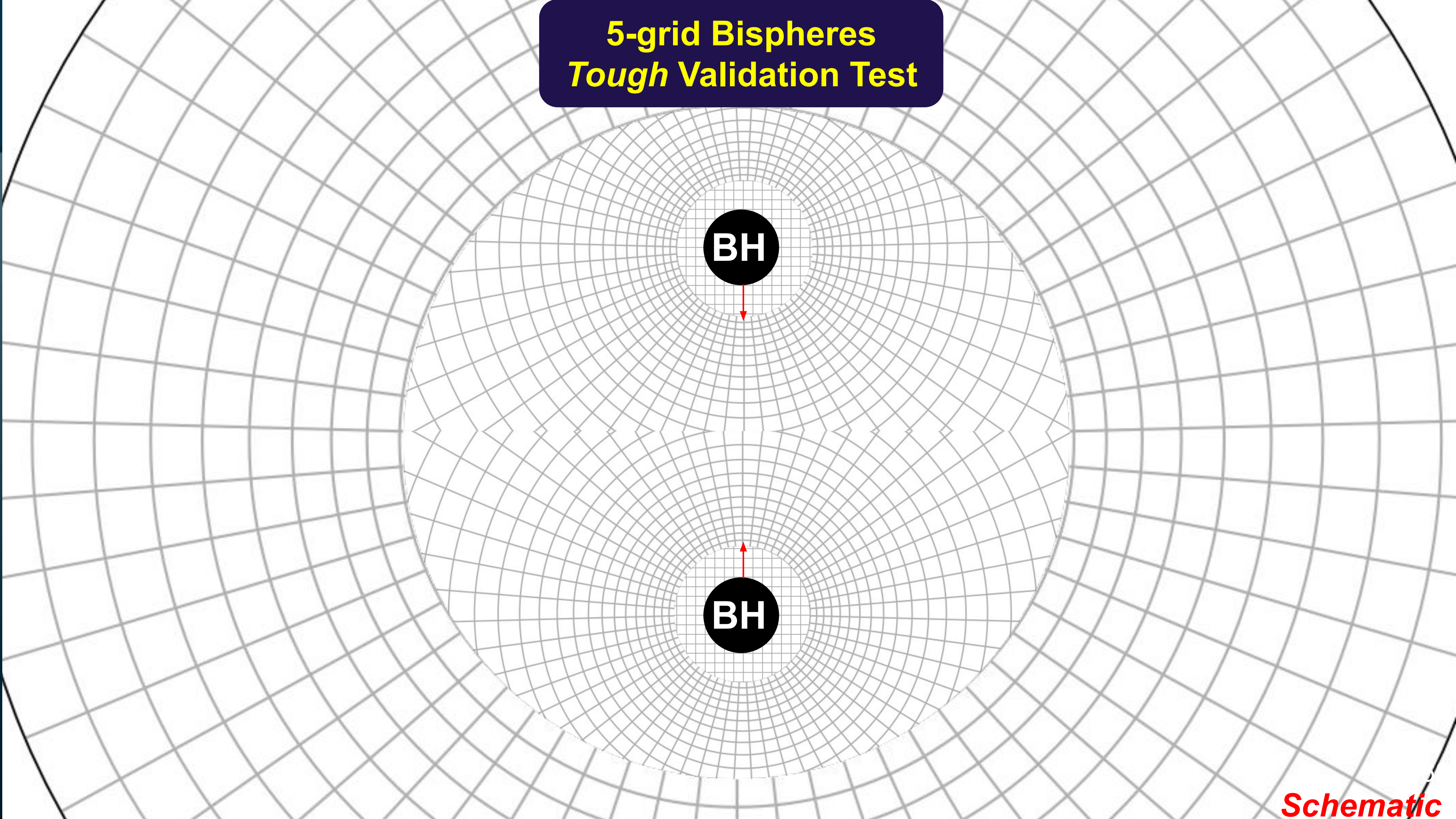

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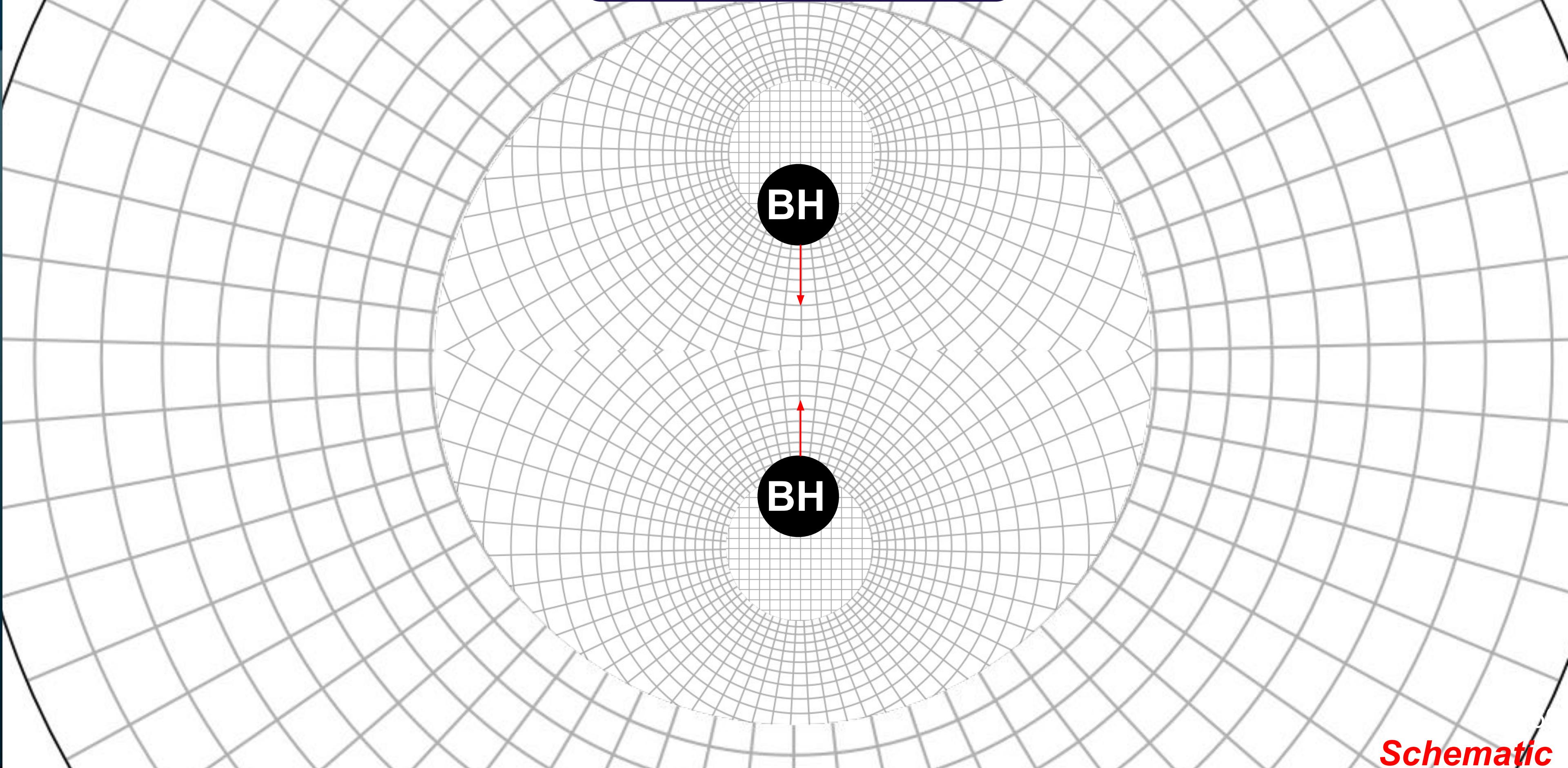

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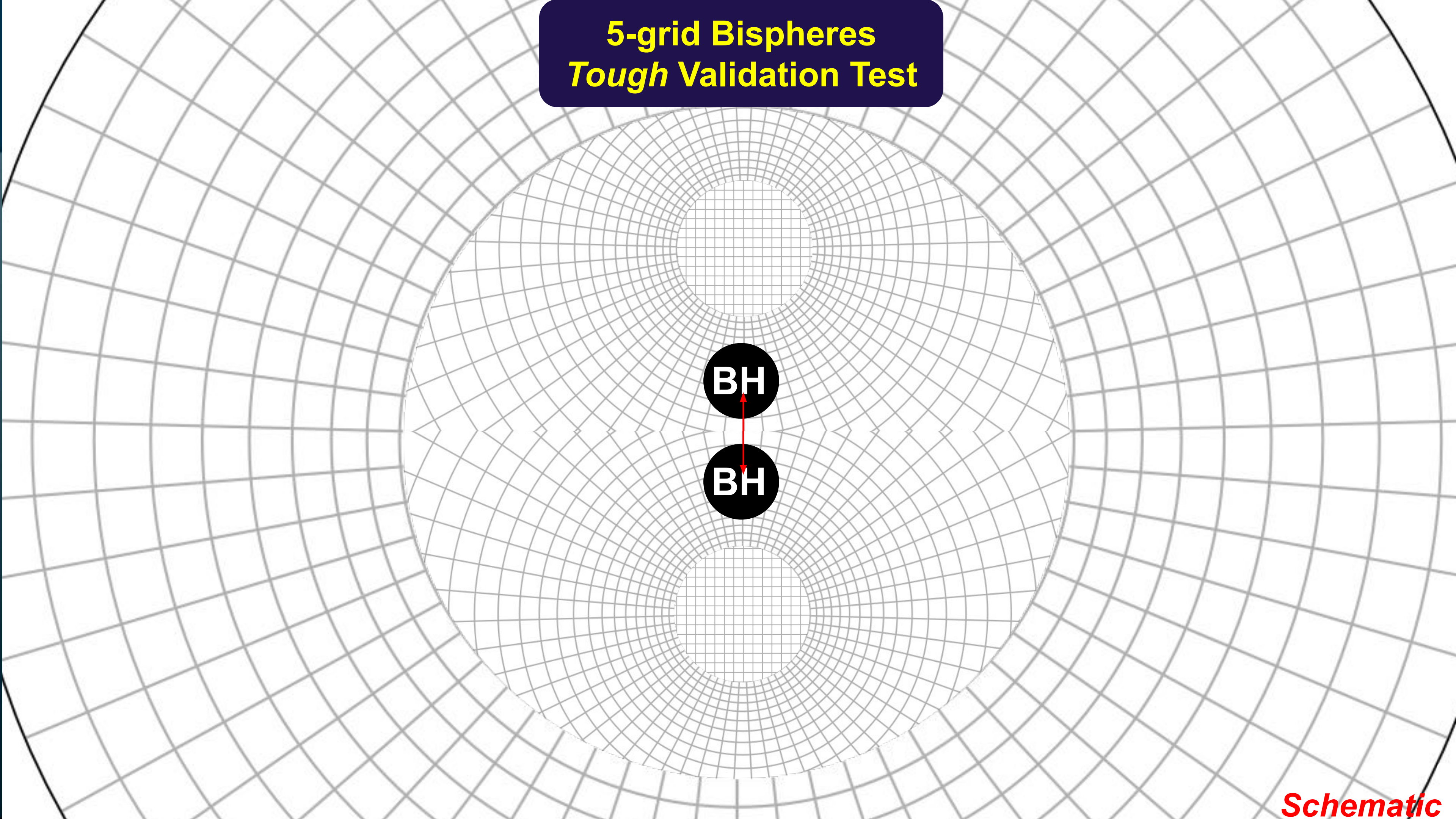

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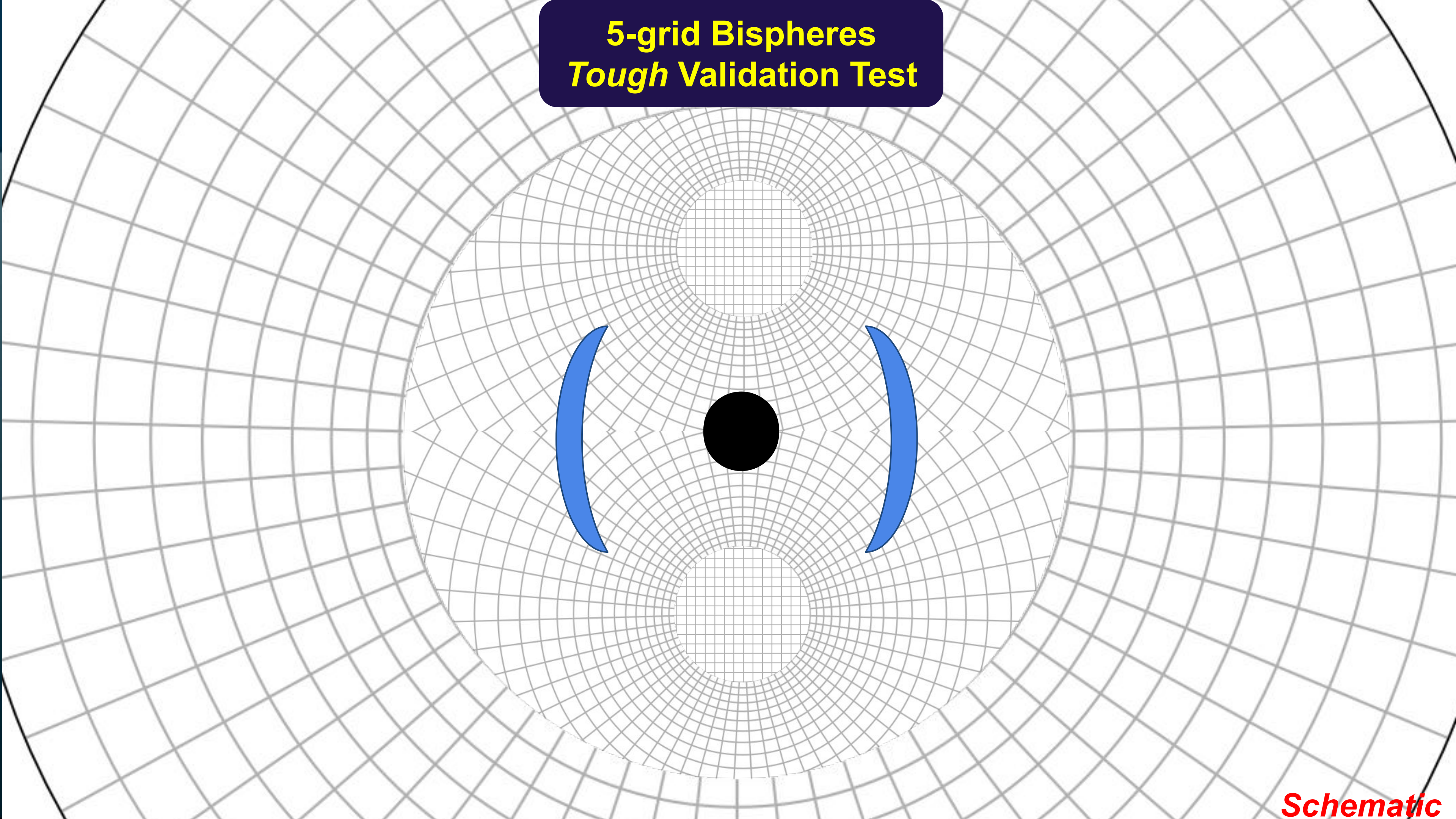
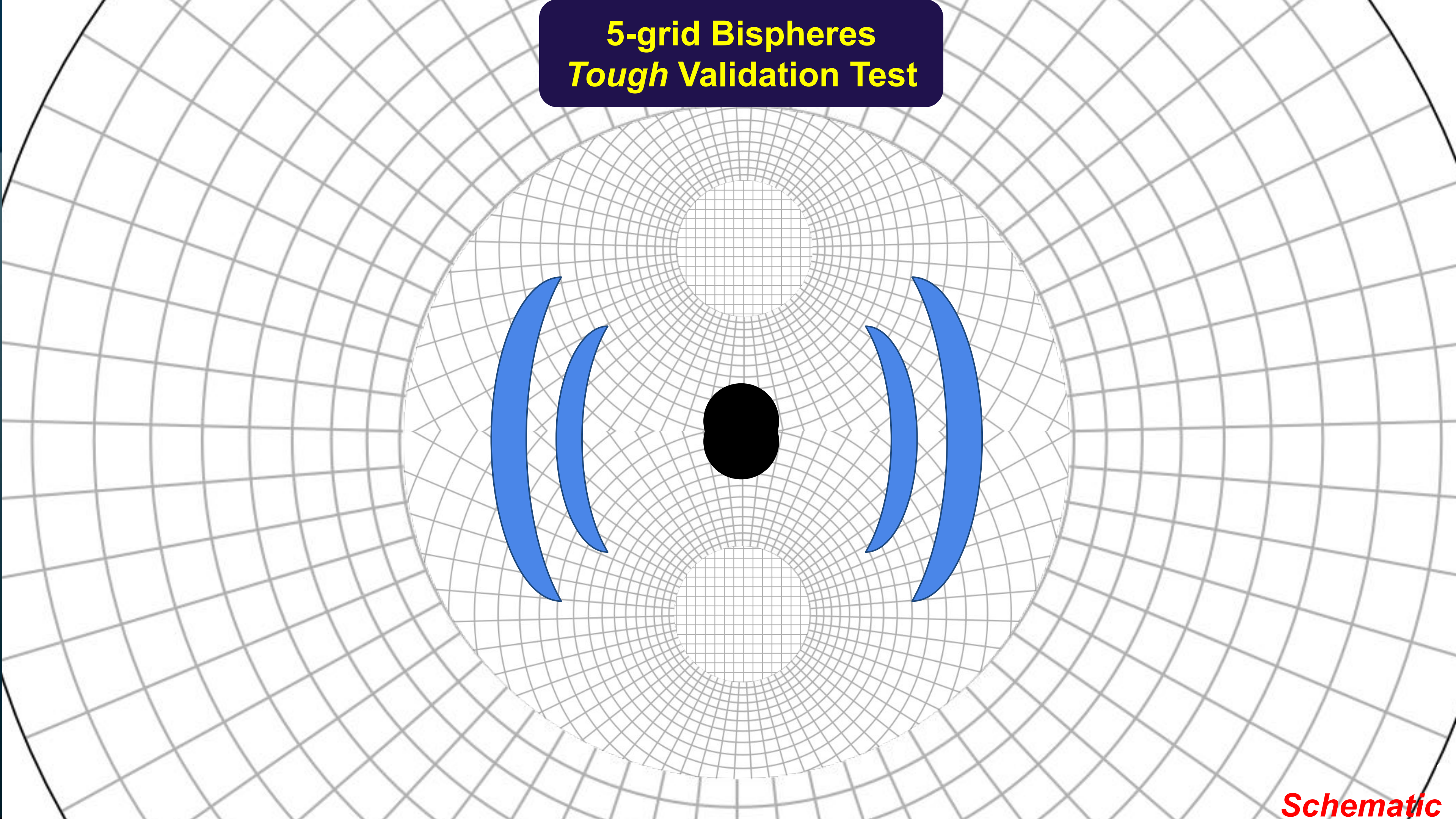


Schematic

5-grid Bispheres *Tough* Validation Test



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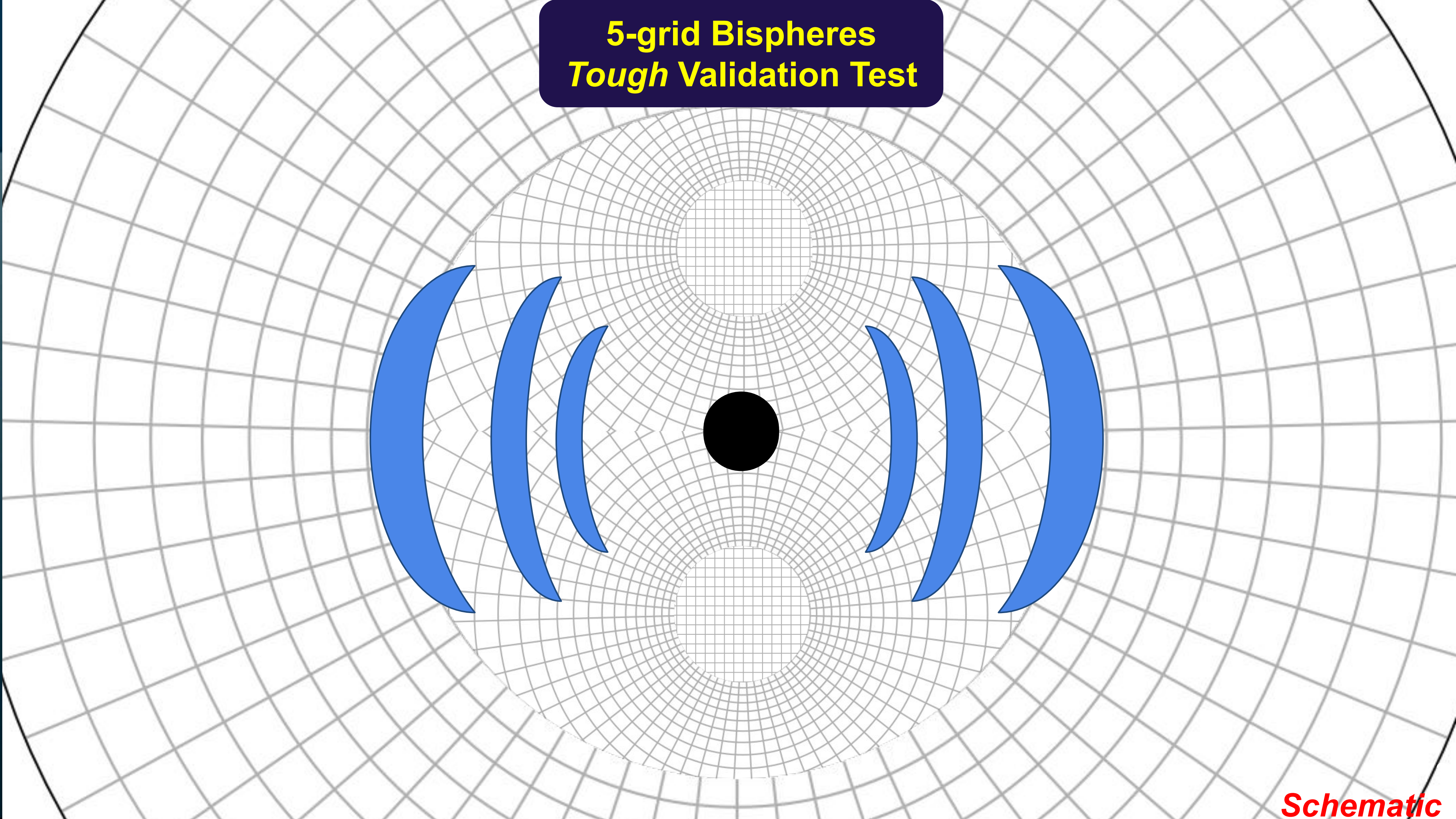

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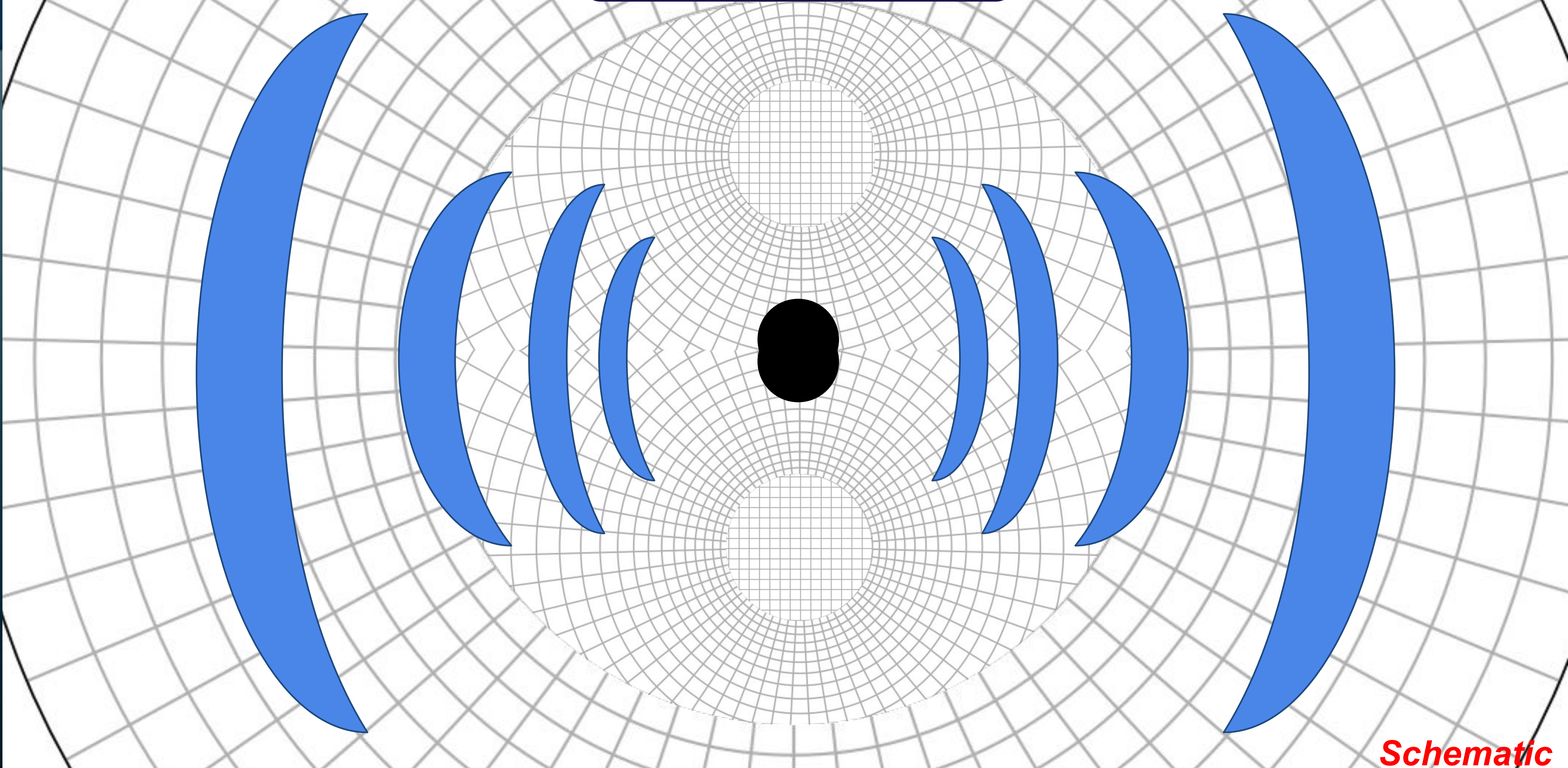
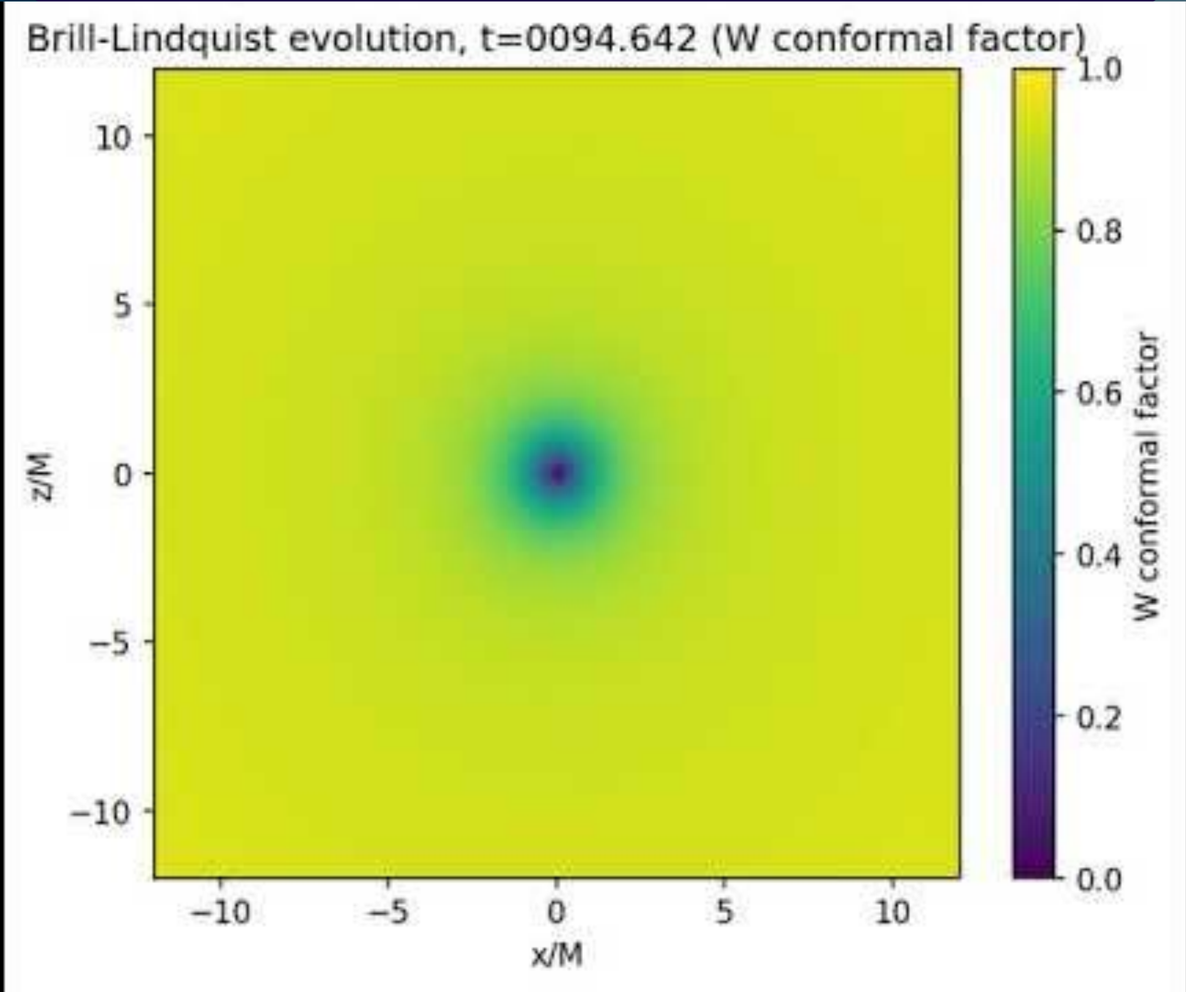


5-grid Bispheres: Two-BH, Head-on Collision



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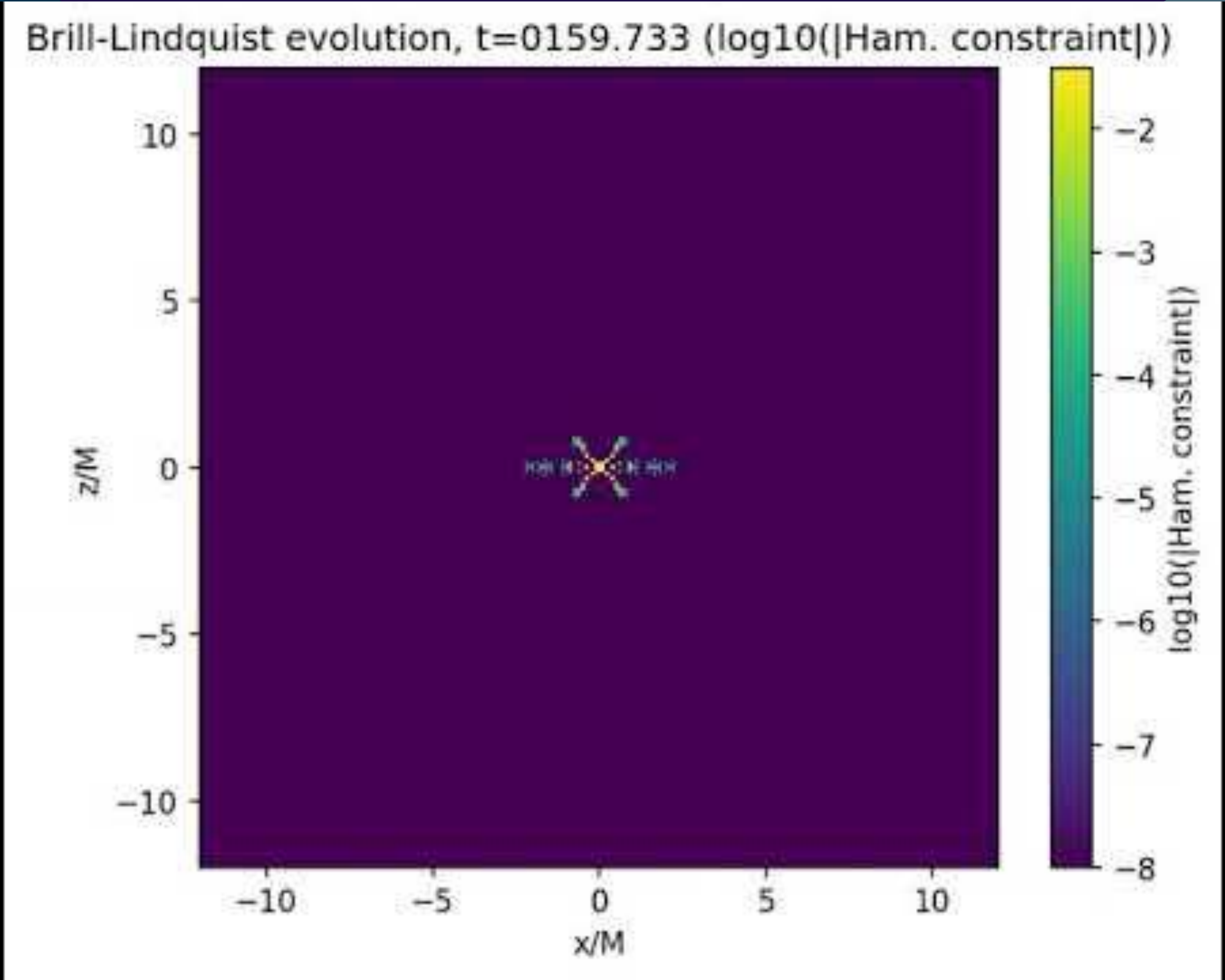

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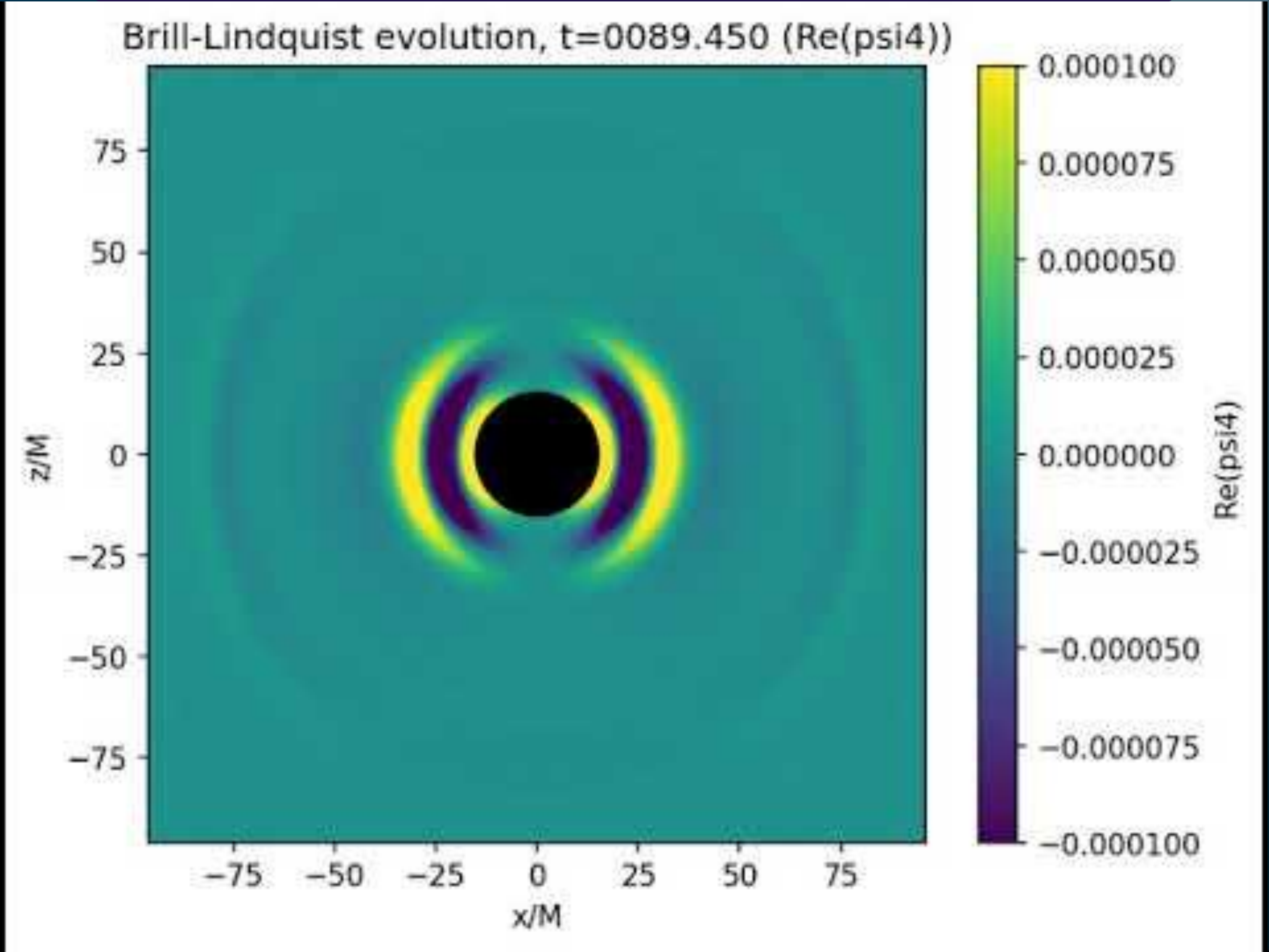
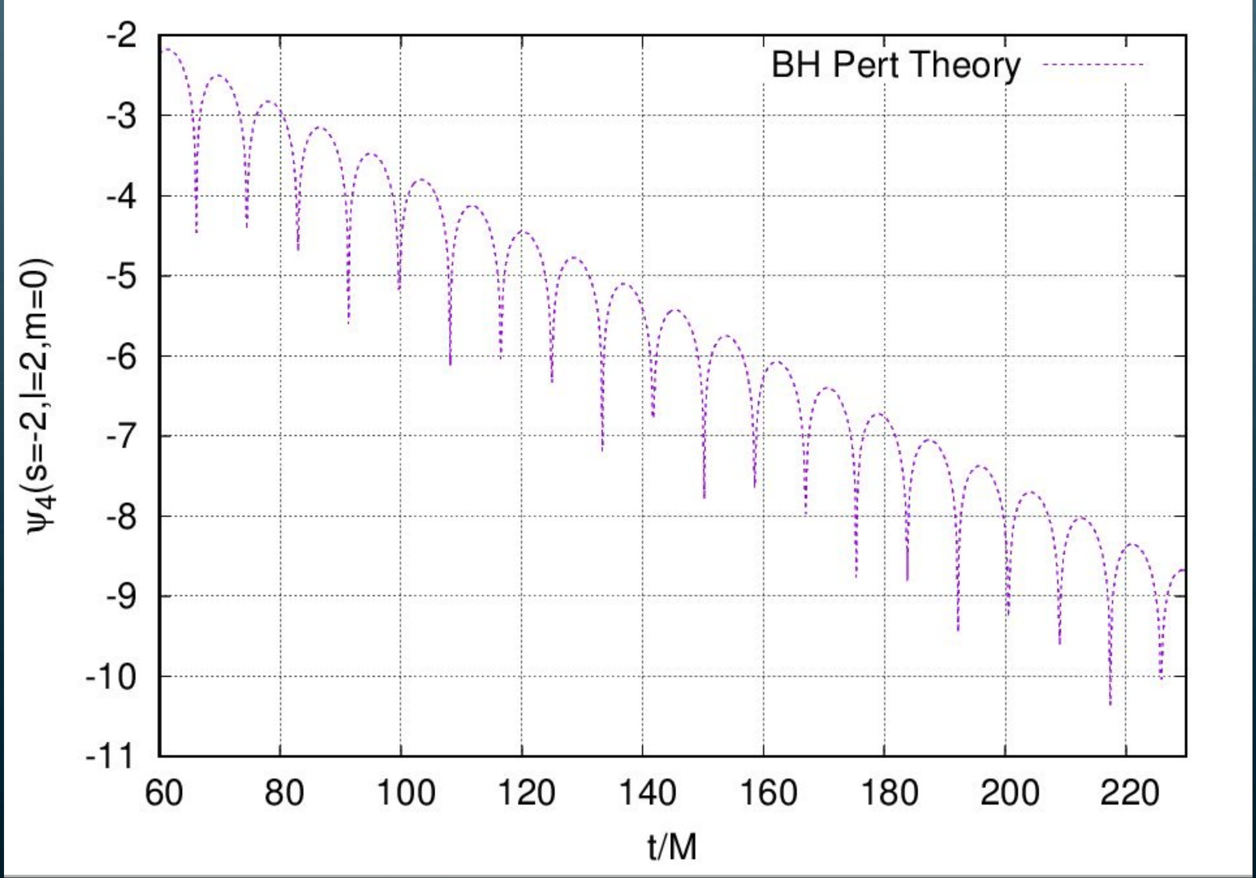


5-grid Bispheres Head-on Results



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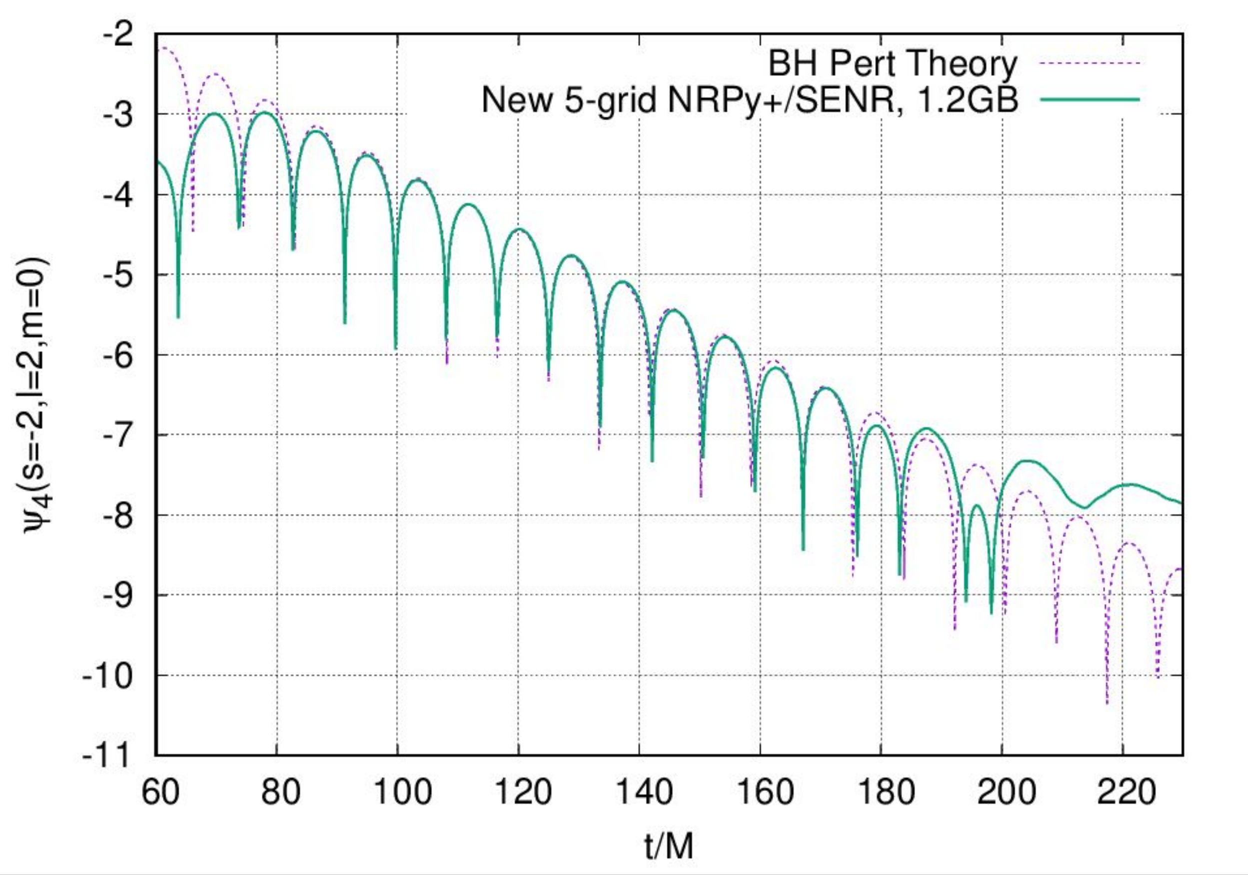

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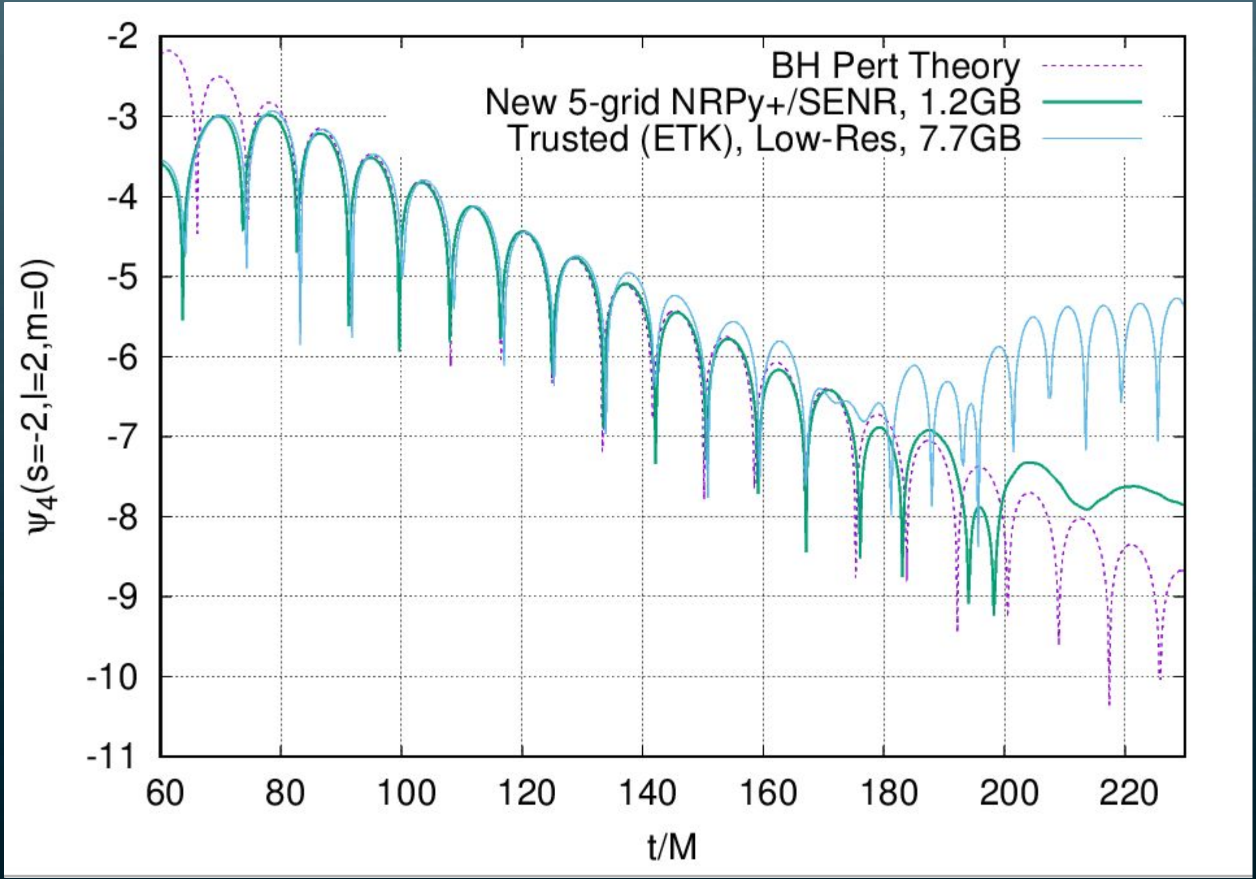

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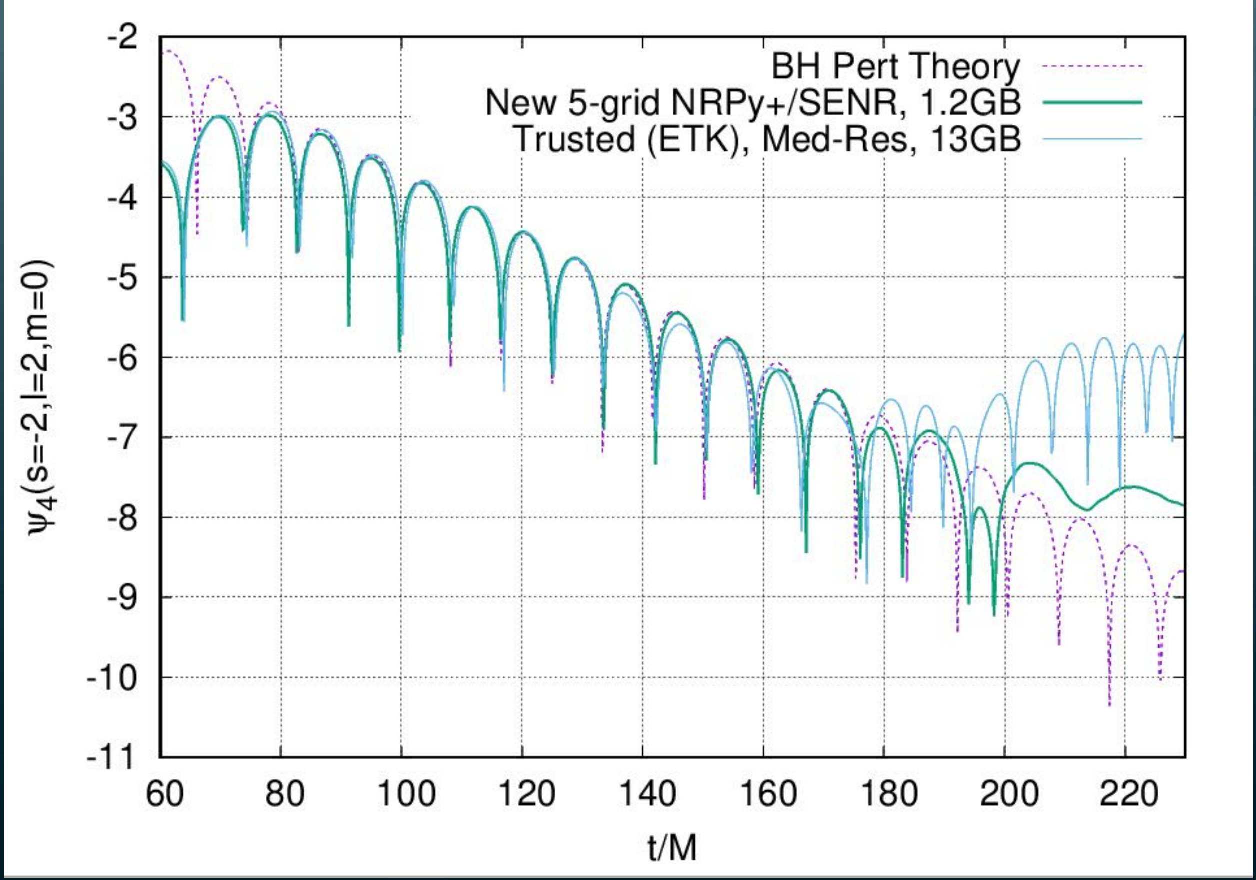

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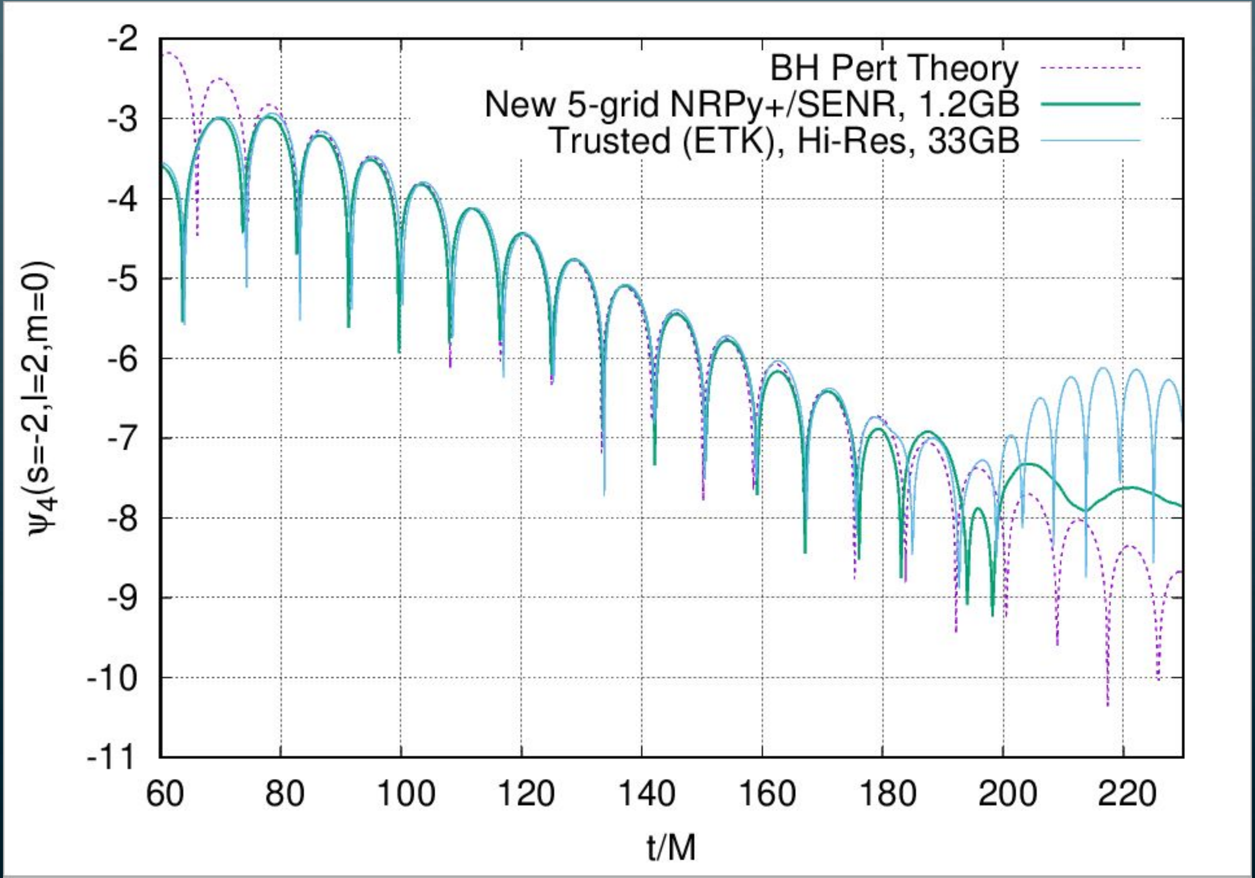

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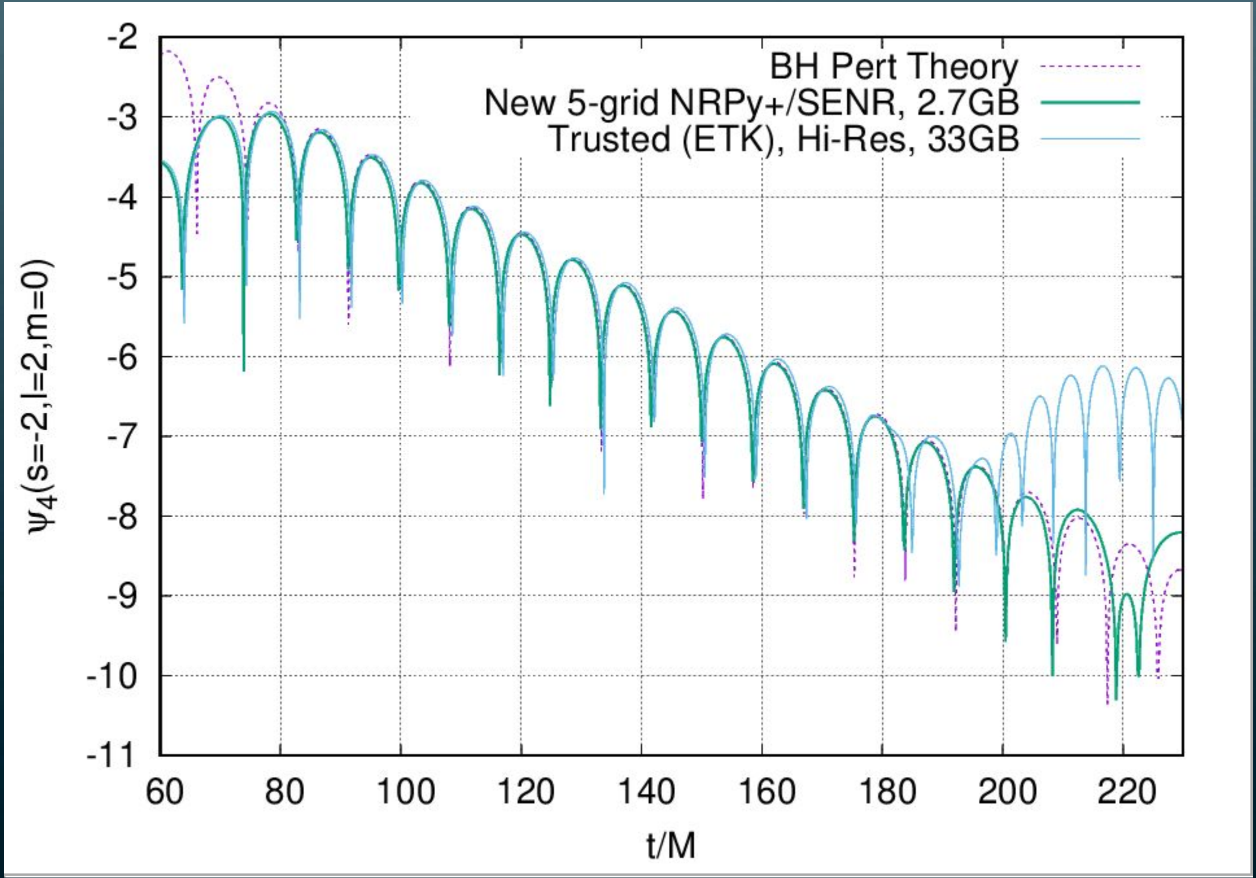

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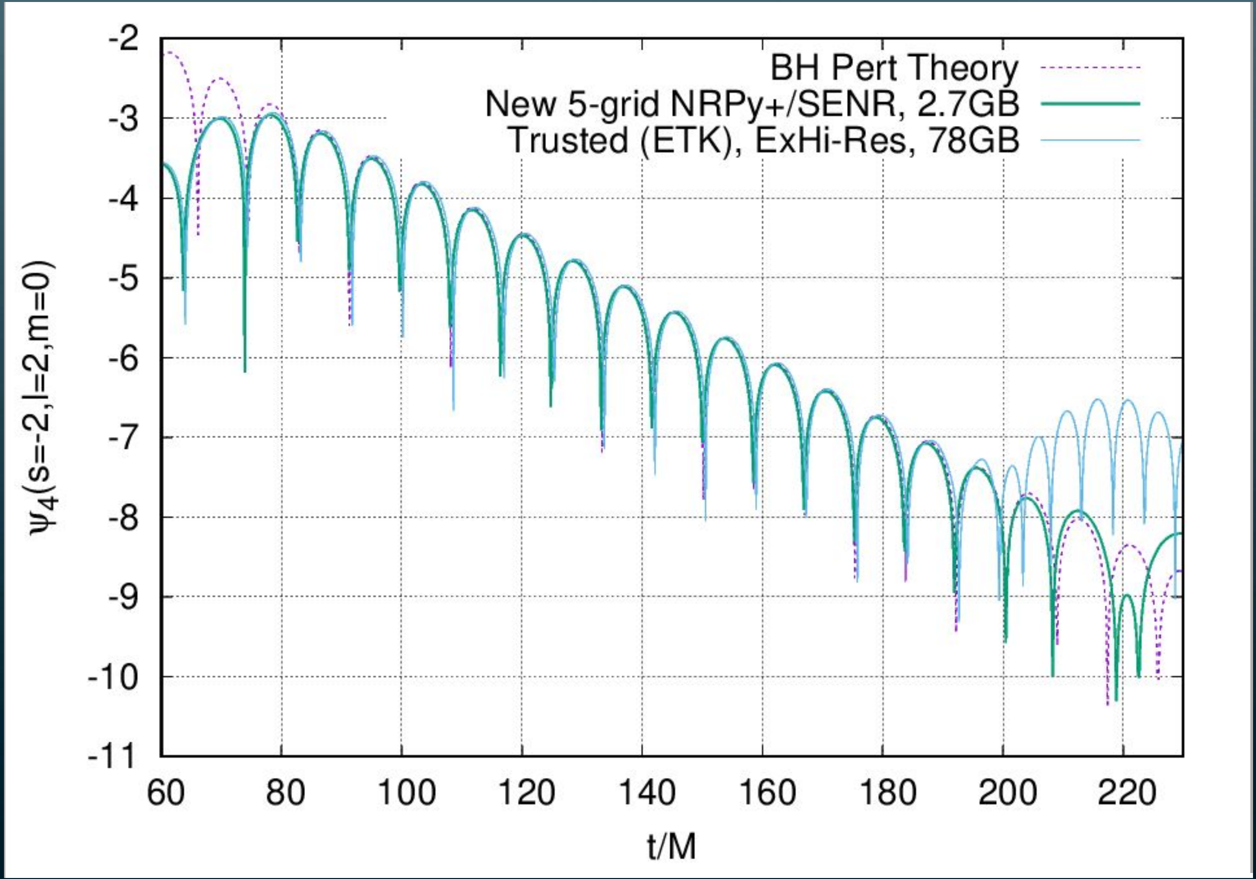
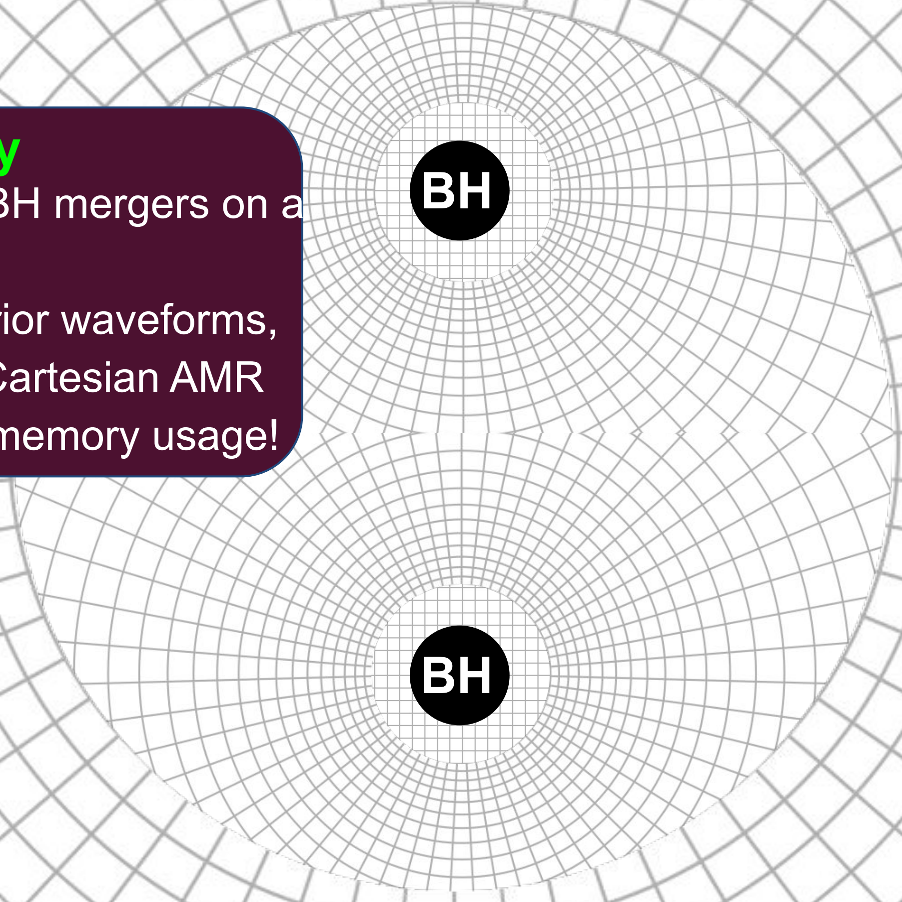


Conclusions

Summary

1. (close-separation) BBH mergers on a cellphone!
2. Comparable or superior waveforms, 5-grid Bispheres vs Cartesian AMR
   a. ~20x less memory usage!



Schematic

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We're so close now!

-={ Last Steps }=-

1. Current focus: regridding algorithm
   a. Hard part (months) was intergrid interpolations!
2. Small modifications:
   a. GW extraction
   b. TwoPunctures ID
3. Finished, not reported here
   a. PN parameters for quasicircular BBH ID
      i. **NRPyPN**

BH

BH

Schematic

Addressing Issues with Singular Coordinates

Baumgarte, Montero, Cordero-Carrión, Müller (PRD 87, 044026, **2012**)

1. **Tensor components can be singular** ($\rightarrow 0$ or ∞) **at coord singularities**
   - Use cell-centered grids to avoid exact overlap with singularities
   - Singular pieces are multiplicative and known analytically:
     1. Scale out singular pieces & handle spatial derivs analytically
     2. Promote rescaled tensors to evolved quantities

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- *Example*: Smooth spacetime quantity Λ^i
  - **Cartesian**: all components regular; no coord singularities

$$\begin{aligned}\bar{\Lambda}^x &= [\text{smooth}] \\ \bar{\Lambda}^y &= [\text{smooth}] \\ \bar{\Lambda}^z &= [\text{smooth}]\end{aligned}$$

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- *Example*: Smooth spacetime quantity Λ^i
  - **Cartesian**: all components regular; no coord singularities
  - **Spherical**: e.g., ϕ component diverges at coord singularity
    - **Idea**: where needed, only take numer. derivatives of smooth part, λ^ϕ
    - Perform *exact* differentiation on singular terms like $1/(r \sin \theta)$

$$\begin{aligned}\bar{\Lambda}^\phi &= \frac{1}{r\sin\theta} \times [\text{smooth part}]\\ &= \frac{1}{r\sin\theta} \times \lambda^\phi\end{aligned}$$

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2. **Divergent multiplicative terms in RHSs of equations**
   - E.g., 1D scalar wave equation:

$$\partial_t^2 u = \partial_r^2 u + \boxed{\frac{2}{r}}\partial_r u$$

   - $\boxed{2/r}$ term "stiffens" the equation
   - Even with cell-centered grids, RK2 timestepping is unstable
     1. Can use PIRK2 (original formulation), but
     2. *Ordinary* RK4 works just fine in 3+1 NR (discovered later)

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***Net result*: Stability & convergence properties *on par* with Cartesian grids**

SENR/NRPy+: Code Validation

http://blackholesathome.net

- Black hole simulation
  - Wormhole initial data
  - Cylindrical coordinates
  - Fourth-order finite differencing

- Excellent convergence
  - at t = 5M, in region unaffected by outer boundary (at r=10M)